Applications of infinite matrices in the theories of orthogonal polynomials and operational calculus

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MVD dec 2014

Infinite matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Pascal matrices coefficients of $(1+x)^n$ in n-th row

They represent translations $f(x) \to f(x+t)$

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They represent translations $f(x) \to f(x + t)$

$\mathscr{P}$ = Polynomials $\mathbb{C}[t]$

$\mathscr{F}_0$ = Formal power series $\sum_{k\geq 0} a_k t^k$

$\mathscr{F}$ = Formal Laurent series $\sum_{k\geq i(a)} a_k t^k$, $i(a) \in \mathbb{Z}$

It is a field.

$\mathscr{L}$ = Lower infinite matrices $A = [a_{j,k}]$, $j,k \geq 0$

$a_{j,k} = 0$ if $j - k < i(A) \in \mathbb{Z}$

$\mathscr{M}$ = Laurent infinite matrices $A = [a_{j,k}]$, $j,k \in \mathbb{Z}$

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(Also called lower (doubly) infinite matrices)

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Elements of $\mathscr{M}$ are of the form

$$\begin{bmatrix} \boldsymbol{A_3} & \boldsymbol{A_2} \\ \boldsymbol{A_4} & \boldsymbol{A_1} \end{bmatrix}$$

$\boldsymbol{A_1} \in \mathscr{L}$, $\boldsymbol{A_2}$ “finite" lower triangular.

$\mathscr{L}$ and $\mathscr{M}$ have very different algebraic properties.

In $\mathscr{L}$ there is a fixed boundary. (End of the world)

Elements of $\mathscr{M}$ are “floating" in $\mathbb{Z} \times \mathbb{Z}$.

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Regular representation

$\sum_{k \geq i(a)} a_k t^k \rightarrow T_a$, Toeplitz matrix

all entries in k-th diagonal of T_a are equal to a_k

position (i,j) is in k-th diagonal if $i-j=k$.

Linear operators

Each matrix A in $\mathscr{M}$ can be seen as a linear operator on $\mathscr{F}$.

Columns can be seen as formal Laurent series and rows as reversed formal Laurent series, (a polynomial in x plus a formal series in x^{-1}).

$\mathscr{M}$ is closed under matrix multiplication.

$\mathscr{M}$ contains multiplication operators (Toeplitz matrices), differential operators, composition operators, change of bases, lowering operators, etc.

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Diagonal representation of matrices

Let S be the matrix that corresponds to multiplication by t.

S is diagonal with $i(S) = 1$, entries $(k+1, k)$ are equal to 1.

We call S the shift matrix. $\{S^k : k \in \mathbb{Z}\}$ is a group isomorphic to $\mathbb{Z}$.

Let $\mathscr{D}_0$ be the set of diagonal matrices of index 0.

It is a commutative algebra isomorphic to the set of sequences $s : \mathbb{Z} \to \mathbb{C}$ with its natural multiplication.

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Given $s : \mathbb{Z} \to \mathbb{C}$ define the matrix $\phi(s)$ in $\mathscr{D}_0$ by $(\phi(s))_{(k,k)} = s(k)$, for $k \in \mathbb{Z}$, and all other entries equal to zero.

Note that $\phi(s)S^m$ is diagonal of index m and $(\phi(s)S^m)_{(j,j-m)} = s(j)$.

Every A in $\mathscr{M}$ can be written as

$$A = \sum_{k \geq i(A)} \phi(s_k)S^k,$$

where each s_k is a function from $\mathbb{Z}$ to $\mathbb{C}$.

$$S^m\phi(s)S^{-m} = \phi(s^{[-m]}), \qquad m \in \mathbb{Z}$$

where $s^{[-m]}$ is defined by $s^{[-m]}(j) = s(j-m)$, for $j \in \mathbb{Z}$.

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Multiplication formula

Let $A = \sum_{k \geq i(A)} \phi(s_k) S^k$ and $B = \sum_{k \geq i(B)} \phi(t_k) S^k$. Then

$$AB = \sum_{n \geq i(A)+i(B)} \left(\sum_k \phi(s_k) \phi(t_{n-k}^{[-k]}) \right) S^n,$$

where k runs over the interval $i(A) \leq k \leq n - i(B)$.

$\mathscr{M}$ is a noncommutative algebra of formal Laurent series with coefficients that are sequences.

It is not easy to characterize the invertible elements of $\mathscr{M}$.

Differentiation operator is $D = \phi(s) S^{-1}$ where $s(k) = k$ for $k \in \mathbb{Z}$.

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Differential operators of the form $\sum_{k=0}^{m} u_k(t)D^k$, where the u_k are polynomials, are represented by banded matrices.

Orthogonality

Let

$$A = I + \sum_{k\geq 1} \phi(s_k)S^k.$$

Then the rows of A are orthogonal with respect to some regular linear functional if and only if there is a matrix

$$L = S^{-1} + \phi(\beta)I + \phi(\alpha)S$$

with $\alpha(k) \neq 0$ for all $k \in \mathbb{Z}$ such that

$$LA = AS^{-1}.$$

Equivalent to three term recurrence relation.

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If $\boldsymbol{C}$ commutes with $\boldsymbol{S}$ or with $\boldsymbol{S}^{-1}$ then $\boldsymbol{C}$ is Toeplitz.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are monic matrices of index zero such that $\boldsymbol{LA} = \boldsymbol{AS}^{-1}$ and $\boldsymbol{LB} = \boldsymbol{BS}^{-1}$ then there is a monic Toeplitz matrix $\boldsymbol{T}$ of index zero such that $\boldsymbol{B} = \boldsymbol{AT}$.

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Some results using matrices in $\mathscr{L}$

Orthogonality, OP as characteristic polynomials of truncated Jacobi matrices,

several characterizations of classical sequences, moments obtained from powers of Jacobi matrix,

explicit formulas for recurrence coefficients of sequences in extended Hahn classes of discrete and q-orthogonal polynomial sequences,

explicit formulas for the Bochner difference and q-difference operators.

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Recurrence coefficients of the extended Hahn class of discrete orthogonal polynomials.

$$\Delta_w f(x) = \frac{f(x+w) - f(x)}{w}.$$

Jacobi matrix $\quad L = S^{-1} + \phi(\beta) I + \phi(\alpha) S$

Define $\sigma_k = \beta_0 + \beta_1 + \cdots + \beta_k$ and $\delta_k = \beta_k - \beta_{k-1}$.

$$g(k) = c_0 k + c_1, \qquad c_0 = \delta_1 - \delta_2, \qquad c_1 = \delta_1 + 5\delta_2.$$

Then we have

$$\sigma_k = (k+1)\left(\delta_0 + \frac{\delta_1 g(3) k}{2g(2k+1)}\right),$$

$$\alpha_k = \frac{k\, g(k-1)}{g(2k)\, g(2k-2)} \times \left(g(2)\alpha_1 - \frac{(k-1)\, g(k)}{4}\left(4t + w^2 - \left(\frac{\delta_1 g(3)}{g(2k-1)}\right)^2\right)\right).$$

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t is a parameter, $t = 0$ for elements in Hahn class.

Note that σ_k (and thus β_k) depends only on $\delta_0, \delta_1, \delta_2$ and k

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$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

$$L^2 D_q - (q+1) L D_q M + q D_q M^2 = t D_q$$

Define

$$g(k) = [k-2]c_0 + c_1, \qquad k \geqslant 0,$$

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$$\sigma_k = [k+1]\frac{c_2 + [k-1]c_3}{g(2k+1)}, \qquad k \geqslant 0.$$

Recurrence coefficients of the extended Hahn class of q-orthogonal polynomials

$$[n] = [n]_q = \frac{1-q^n}{1-q} \text{ for } n \in \mathbb{Z}.$$

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

$$L^2 D_q - (q+1) L D_q M + q D_q M^2 = t D_q$$

Define

$$g(k) = [k-2]c_0 + c_1, \qquad k \geqslant 0,$$

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where

$$F_k = \frac{q^{k-3}((q^{k-1}+1)\sigma_0 c_0 - c_3)(q^{k-2}(\sigma_0 c_0 - c_3) + \sigma_0 c_4)}{g(2k-1)^2} - t.$$

Note that σ_k depends on $\sigma_0, \sigma_1, \sigma_2, q$, and k.

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For each class we have a uniform parametrization in terms of the "initial" recurrence coefficients $\beta_0, \beta_1, \beta_2, \alpha_1$.

There are special cases such as $\delta_1 = \delta_2 = 0$, and cases that yield some $\alpha_k = 0$

The union of the extended Hahn classes includes the following polynomial sequences:

Askey-Wilson, q-Racah, continuous dual q-Hahn, big q-Jacobi, little q-Laguerre,

Wall, Stieltjes-Wiegart, Charlier, Meixner, continuous Hahn, Hahn, Krawtchuk,

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Are all sequences in the Askey scheme included ?

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Some examples of discrete orthogonal sequences

Simple case $\beta_k = \beta_0$

(1) $\alpha_n = \alpha_1 n^2$

(2) $\alpha_n = \alpha_1 \binom{n}{2}$

(3)
$$\alpha_n = \alpha_1 \frac{n(n-1+d)}{d}, \qquad d > 0.$$

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Example of a q-orthogonal sequence

$$\sigma_n = \frac{(q^{n+2}-1)(q^{n+1}-1)\beta_0}{(q^2-1)(q-1)}$$

$$\alpha_n = \beta_0^2 \frac{(q^n-1)q^{2n}(q^{n+1}-1)}{(q-1)^2(q+1)^2}$$

Change of pseudo-bases

Instead of $\{t^k : k \in \mathbb{Z}\}$ we can use a set $G = \{p_k : k \in \mathbb{Z}\}$ of suitable objects (functions)

and define a multiplication in $\mathscr{F}$ by defining $p_k p_m = p_{k+m}$, which is not necessarily the natural multiplication of the p_k as functions.
Then G becomes a cyclic group isomorphic to $\mathbb{Z}$.

Examples
If $p_k = t^k/k!$ we have $(t^k/k!)(t^m/m!) = t^{k+m}/(k+m)!$ (Convolution).

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The modified left shift

Shift operators

$$S : \mathscr{F} \to \mathscr{F}, \qquad Sa = p_1 a, \qquad \text{Right shift}$$

$$S^{-1} : \mathscr{F} \to \mathscr{F}, \qquad S^{-1} a = p_{-1} a, \qquad \text{Left shift}$$

S generates a group isomorphic to $\{p_k : k \in \mathbb{Z}\}$.

Modified left shift

Define $L : \mathscr{F} \to \mathscr{F}$, by

$$Lp_0 = 0,$$

and

$$Lp_k = p_{k-1}, \qquad \text{if} \quad k \neq 0, \quad k \in \mathbb{Z}.$$

L is not invertible, $\quad \text{Ker}(L) = \langle p_0 \rangle$.

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Define the projections $\quad P_m : \mathscr{F} \to \langle p_m \rangle \quad$ by

$$P_m a = P_m \left(\sum_{k \geq i(a)} a_k p_k \right) = a_m p_m, \qquad m \in \mathbb{Z}.$$

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$$L = S^{-1}(I - P_0) \quad \text{and} \quad S^k P_n S^{-k} = P_{n+k}.$$

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Solve $\quad w(L)f = g$

where w is a polynomial, $g \in \mathscr{F}$, f is the unknown.

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A simple case; $L - xI$, $x \in \mathbb{C}$

$$L - xI = S^{-1}(I - P_0) - x\,S^{-1}S = S^{-1}(I - xS - P_0)$$

$$f \in \mathrm{Ker}(L - xI) \iff (I - xS)f = P_0 f \iff$$

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Notation

The geometric series associated with x is

$$e_{x,0} = \frac{p_0}{p_0 - x p_1} = \sum_{n \geq 0} x^n p_n, \qquad x \in \mathbb{C}$$

Then $\quad \mathrm{Ker}(L - xI) = \langle e_{x,0} \rangle.$

The image of $L - xI$

$$g = (L - xI)f = S^{-1}(I - xS - P_0)f = p_{-1}((p_0 - xp_1)f - f_0 p_0).$$

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Particular solution

If $g \in \mathrm{Im}(L - xI)$ then

$$(L - xI)p_1 e_{x,0} g = g.$$

Therefore $p_1 e_{x,0} g$ is a particular solution of $(L - xI)f = g$.

Goal: **Solve $w(L)f = g$**

Find a basis for the Kernel of $w(L)$,

a characterization for the Image of $w(L)$,

and an explicit construction of a particular solution of $w(L)f = g$.

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Multiplication of geometric series

$$p_1\, e_{x,0}\, e_{y,0} = \frac{e_{x,0} - e_{y,0}}{x - y}, \qquad x \neq y.$$

Divided differences

Shifted powers of geometric series

For $(x,k) \in \mathbb{C} \times \mathbb{N}$ define

$$e_{x,k} = \frac{p_k}{(p_0 - x p_1)^{k+1}} = p_k \left(e_{x,0}\right)^{k+1} = \sum_{n \geq k} \binom{n}{k} x^{n-k} p_n.$$

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Multiplication and shift ∼ differentiation with respect to x

$$e_{x,k} = \frac{D_x^k}{k!} e_{x,0}, \qquad p_1 e_{x,0} e_{x,0} = e_{x,1}.$$

Divided differences

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Using the Leibniz rule we get a linear combination of $e_{x,j}$, $e_{y,i}$.

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The basic results

Let w be a polynomial of degree $n+1$

$$w(t) = \prod_{k=0}^{r} (t - x_k)^{m_k+1}$$

Define $$w(L) = \prod_{k=0}^{r} (L - x_k I)^{m_k+1}.$$

Kernel and Image of $w(L)$

$$\mathrm{Ker}(w(L)) = \langle e_{x_k,i} : 0 \le k \le r, \ 0 \le i \le m_k \rangle.$$

There exists a unique $d_w \in \mathscr{F}$ such that

$$\mathrm{Im}(w(L)) = \{ g \in \mathscr{F} : d_w g \in \mathrm{Ker}(P_0 + P_1 + \cdots + P_n) \}.$$

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Theorem

Let $\boldsymbol{g} \in \mathbf{Im}(\boldsymbol{w}(\boldsymbol{L}))$. Then $\boldsymbol{w}(\boldsymbol{L})\boldsymbol{d}_{\boldsymbol{w}}\boldsymbol{g} = \boldsymbol{g}$,

$$\{\boldsymbol{f} \in \mathscr{F} : \boldsymbol{w}(\boldsymbol{L})\boldsymbol{f} = \boldsymbol{g}\} = \{\boldsymbol{d}_{\boldsymbol{w}}\boldsymbol{g} + \boldsymbol{h} : \boldsymbol{h} \in \mathbf{Ker}(\boldsymbol{w}(\boldsymbol{L}))\},$$

and $\boldsymbol{d}_{\boldsymbol{w}} = \boldsymbol{p}_{r+1}\, \boldsymbol{e}_{x_0,m_0}\, \boldsymbol{e}_{x_1,m_1} \cdots \boldsymbol{e}_{x_r,m_r}$

$\boldsymbol{w}(\boldsymbol{L})\boldsymbol{d}_{\boldsymbol{w}} = \boldsymbol{p}_0$, $\boldsymbol{d}_{\boldsymbol{w}}$ is "right inverse" of $\boldsymbol{w}(\boldsymbol{L})$

$\boldsymbol{d}_{\boldsymbol{w}}\boldsymbol{g}$ has all its "initial values" equal to zero

Dimension of $\mathbf{Ker}(\boldsymbol{w}(\boldsymbol{L}))$ is $\boldsymbol{n}+\mathbf{1}$ = degree of $\boldsymbol{w}$

For any given initial values there is a unique $\boldsymbol{h} \in \mathbf{Ker}(\boldsymbol{w}(\boldsymbol{L}))$ with those initial values. Hermite interpolation problem.

Theorem

Let $g \in \mathrm{Im}(w(L))$. Then $w(L)d_w g = g$,

$$\{f \in \mathscr{F} : w(L)f = g\} = \{d_w g + h : h \in \mathrm{Ker}(w(L))\},$$

and $$d_w = p_{r+1}\, e_{x_0,m_0}\, e_{x_1,m_1} \cdots e_{x_r,m_r}$$

$w(L)d_w = p_0$, d_w is “right inverse" of $w(L)$

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Differential equations

Generators

Let t be a complex variable. Define

$$p_k = \frac{t^k}{k!}, \qquad k \in \mathbb{Z},$$

where

$$k! = \frac{(-1)^{-k-1}}{(-k-1)!}, \qquad k < 0.$$

Modified left shift

Since $D_t p_k = p_{k-1}$ for $k \neq 0$ and $D_t p_0 = 0$, the operator D_t is the modified left shift.

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Geometric series

$$e_{x,0} = \sum_{k \geq 0} x^k \frac{t^k}{k!} = \exp(xt).$$

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$$e_{x,m} = \frac{D_x^m}{m!} \exp(xt) = \frac{t^m}{m!} \exp(xt).$$

Exponential polynomials

The subalgebra

$$\langle e_{x,m} : (x,m) \in \mathbb{C} \times \mathbb{N} \rangle$$

of $\mathscr{F}$ is the space of exponential polynomials, also called pseudo-polynomials.

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Let t be a complex variable and

$$p_k = \frac{(\log t)^k}{k!}, \qquad k \in \mathbb{Z},$$

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$$L = tD$$

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$$e_{x,0} = \sum_{k \geq 0} x^k \frac{(\log t)^k}{k!} = \exp(x \log t) = t^x.$$

and

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Euler's equation (Mellin's transform)

$$(t^2 D^2 + 4tD + 2I) f(t) = \exp(-t)$$

becomes

$$(L+I)(L+2I) f = g = \sum_{k \geq 0} \frac{(-1)^k}{k!} e_{k,0}.$$

$$d_W g = p_1 (e_{-1,0} - e_{-2,0}) g = t^{-2} \exp(-t) - t^{-2} + (\exp(-1) - 1) t^{-1}.$$

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Some concrete realizations

L	p_n	$e_{x,0}$
D_t	$t^n/n!$	$\exp(xt)$
Δ	$\binom{k}{n}$	$(1+x)^k$
tD_t	$(\log t)^n/n!$	t^x
$a(t)D_t + b(t)I$	$\exp(-u(t))v^n(t)/n!$	$\exp(xv(t) - u(t))$

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