# Orthogonal polynomials and integral transforms

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# Outline

Boas-Buck type polynomials · (d)-Orthogonal polynomials

Classical polynomials · Semiclassical polynomials

Index Integral transforms · Kontorovich-Lebedev transform

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A Monic Orthogonal Polynomial Sequence (MOPS) $\{P_n\}_{n\geqslant 0}$ is defined by

$$\langle u_0, P_n P_k \rangle = N_n \delta_{n,k} \ , \ \text{with} \ \ N_n \neq 0.$$

where $u_0$ is the first element of the corresponding dual sequence (canonical form).

- In this case $u_0$ is said to be **regular** (or quasi-definite).

- It always satisfies the second order recurrence relation

$$P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x)$$

with $P_0 = 1$ and $P_{-1} = 0$ and

$$\beta_n = \frac{\langle u_0, x P_n^2 \rangle}{\langle u_0, P_n^2 \rangle} \qquad \text{and} \qquad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2 \rangle}{\langle u_0, P_n^2 \rangle} \neq 0, \ n \in \mathbb{N}$$

# The $d$-orthogonality

**Definition.** A MPS $\{P_n\}_{n\geqslant 0}$ is $d$-orthogonal with respect to the vector functional $\mathbf{U} = (u_0, \ldots, u_{d-1})^T$, iff ... (Maroni,1989)(van Iseghem,1987)

$$\begin{cases} \langle u_k, x^m P_n \rangle = 0 & , \quad n \geqslant md + k + 1 \ , \ m \geqslant 0, \\ \langle u_k, x^m P_{md+k} \rangle \neq 0 & , \quad m \geqslant 0. \end{cases} \tag{1}$$

and $k = 0, 1, \ldots, d-1$.

In this case, the $d$-MOPS $\{P_n\}_{n\geqslant 0}$ necessarily satisfies the $(d+1)$-order recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \sum_{\nu=0}^{d-1} \gamma_{n-\nu}^{d-1-\nu} P_{n-1-\nu}(x) \quad , \quad n \geqslant d+1, \tag{2}$$

where $\gamma_{n+1}^0 \neq 0$ for all $n \geqslant 0$, and $\gamma_k^{-m} = 0$, for $m \geqslant 0$.

## Classical polynomials

**Definition.** A MOPS $\{P_n\}_{n\geqslant 0}$ is classical iff $\{\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}P_{n+1}(x)\}_{n\geqslant 0}$ is also an MOPS.

For any MOPS $\{P_n\}_{n\geqslant 0}$ the following statements are equivalent.

(a) $\{P_n\}_{n\geqslant 0}$ is classical

(b) There exists a pair of polynomials $(\phi, \psi)$ such that

$$D(\phi u_0) + \psi u_0 = 0$$

where $\max(\deg \phi - 2, \deg \psi - 1) = 0$ with $\phi$ monic.

(b) There exists a pair of polynomials $(\phi, \psi)$ and a sequence $\{\lambda_n \neq 0\}_{n\geqslant 0}$ such that

$$\phi(x)P''_{n+1}(x) - \psi(x)P'_{n+1}(x) = \lambda_n P_{n+1}(x)\ ,\ \ n \geqslant 0$$

(c) There exists a monic polynomial $\phi$ with $\deg \phi \leqslant 2$ such that

$$\Phi(x)P'_{n+1}(x) = \sum_{\nu=n}^{n+\deg\Phi} \theta_{n,\nu}P_\nu(x) \qquad \text{with} \qquad \theta_{n,n}\theta_{n,n+t} \neq 0, \quad n \geqslant 0.$$

(d) There exists a monic polynomial $\phi$ with $\deg \phi \leqslant 2$ and a sequence $\{\theta_n \neq 0\}_{n\geqslant 0}$ such that $\quad P_n u_0 = \theta_n D^n(\phi^n u_0)\ ,\ \ n \geqslant 0.$

## Classical polynomials

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The **$\mathcal{O}$-classical polynomials** $\{P_n\}_{n\geqslant 0}$ (orthogonal with respect to $u_0$)

- $\{P_n^{[1]}\}_{n\geqslant 0}$ is also orthogonal, where $P_n^{[1]}(x) = \frac{1}{\rho_{n+1}}\,\mathcal{O}P_{n+1}(x)$
  (Hahn's property)

- ${}^t\mathcal{O}(\phi u_0) + \psi u_0 = 0$ with $\deg \Phi \leqslant 2$ and $\deg \Psi = 1$

Examples for the operator $\mathcal{O}$ :

- $\mathcal{O} = D$ : Hermite, Laguerre, Bessel and Jacobi
- $\mathcal{O} = D_q$ where $D_q f(x) := \frac{f(qx)-f(x)}{(q-1)x}$ (Khérigi & Maroni, 2001)
- $\mathcal{O} = \Delta_\omega$ where $\Delta_\omega f(x) := \frac{f(x+\omega)-f(x)}{\omega}$ (Abdelkharim & Maroni, 1997)
- $\mathcal{O} = D + \theta D_{-1}$ (Ben Cheikh, 2009)
- $\ldots$

# Semiclassical polynomials

**Definition.** A MOPS $\{P_n\}_{n\geqslant 0} \perp u$ is called **semiclassical** when $\exists \Phi, \Psi \in \mathcal{P}$, with $\Phi$ monic and $\deg \Psi \geqslant 1$, such that

$$(\Phi u)' + \Psi u = 0. \tag{3}$$

The pair $(\Phi, \Psi)$ is not unique.

more generally...

$\{P_n\}_{n\geqslant 0}$ is $\mathcal{O}$-semiclassical, whenever the corresponding regular form $u_0$ fulfils

$${}^t\mathcal{O}(\Phi u_0) + \Psi u_0 = 0$$

with $\deg \Phi = t \geqslant 0$ and $\deg \Psi = p \geqslant 1$ .

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## The semiclassical polynomials

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- $^t\mathcal{O} = D$ :
  The recurrence coefficients of $D$-semiclassical polynomial sequences are often related to Painlevé type equations.

  Magnus (1995,1999), Clarkson (2008), Chen & Its (2010), Chen & Zhang (2010), Dai & Zhang (2010), Clarkson & Jordaan (2013), etc...
- $^t\mathcal{O} = \Delta_\omega$ : (Maroni & Mejri, 2008), where the symmetric case is treated for the class $s = 1$.

  - connections to discrete Painlevé type equations : (Boelen, Filipuk &Van Assche (2011,2012) ), Clarkson & Jordaan (2013), etc.
- $^t\mathcal{O} = D_q$ , we refer to (Khériji, 2003), (Ghressi & Khériji, 2009), (Mejri, 2009), (Ormerod, Witte & Forrester, 2011) , (Boelen, Smet & Van Assche, 2010)

# Index integral transforms

In 1964, Wimp formally introduced the general index transform over parameters of the Meijer $G$-function

$$F(\tau) = \int_0^\infty G^{m,n+2}_{p+2,q}\left(x; \begin{matrix} 1-\mu+i\tau, 1-\mu-i\tau, (a_p) \\ (b_q) \end{matrix}\right) f(x)dx \ ,$$

whose inversion formula

$$f(x) = \frac{1}{\pi^2}\int_0^\infty \tau \sinh(2\pi\tau) F(\tau)\, G^{q-m,p-n+2}_{p+2,q}\left(x; \begin{matrix} \mu+i\tau, \mu-i\tau, -(a_p^{n+1}), -(a_n) \\ -(b_q^{m+1}), -(b_m) \end{matrix}\right) d\tau \ .$$

was established in 1985 by Yakubovich.

**Examples.**
Kontorovich-Lebedev (KL) · Mehler-Fock · Olevski-Fourier-Jacobi · Whittaker ·

# The Kontorovich-Lebedev (KL) transform

For $\alpha > 0$, consider

$$KL_\alpha[f](\tau) = 2\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^{-2}\int_0^\infty x^\alpha K_{i\tau}(2\sqrt{x})f(x)dx\ ,$$

$$x^{\alpha+1}\ f(x) = \frac{1}{\pi^2}\lim_{\lambda\to\pi-}\int_0^\infty \tau\sinh(\lambda\tau)\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^2 K_{i\tau}(2\sqrt{x})\ KL_\alpha[f](\tau)d\tau\ .$$

valid for any continuous function $f \in L_1\left(\mathbb{R}_+, K_0(2\mu\sqrt{x})dx\right)$, $0<\mu<1$, in a neighborhood of each $x \in \mathbb{R}_+$ where $f(x)$ has bounded variation.

Thus,

$$KL_\alpha : \quad x^n \quad \longmapsto \quad (\alpha+1-\tfrac{i\tau}{2})_n(\alpha+1+\tfrac{i\tau}{2})_n = \prod_{\sigma=1}^{n}\left((\alpha+1+\sigma)^2+\tfrac{\tau^2}{4}\right)$$

where $(a)_n := a(a+1)\dots(a+n-1)$

# a change of basis...

$$\left(\frac{\tau^2}{4}+\alpha^2\right)^n = \sum_{\nu=0}^{n} T_{n,\nu}(\alpha)\left|\left(\alpha+1+\frac{i\tau}{2}\right)_\nu\right|^2$$

whereas (4)

$$\left|\left(\alpha+1+\frac{i\tau}{2}\right)_n\right|^2 = \sum_{\nu=0}^{n} t_{n,\nu}(\alpha)\left(\frac{\tau^2}{4}+\alpha^2\right)^\nu \ , \ n \geqslant 0.$$

$\{t_{n,\nu}(\alpha), T_{n,\nu}(\alpha)\}$ are essentially the non-centered central factorial numbers defined by the triangular recurrence relations

$$t_{n,n}(\alpha)=1 \quad ; \quad t_{n+1,\nu}(\alpha)=t_{n,\nu-1}(\alpha)+(2\alpha+n+1)(n+1)\ t_{n,\nu}(\alpha)\ ,\ 0\leqslant\nu\leqslant n,$$

with $t_{n,-1}(\alpha)=0$, while

$$T_{n,n}(\alpha)=1 \quad ; \quad T_{n+1,\nu}(\alpha)=T_{n,\nu-1}(\alpha)-(2\alpha+\nu+1)(\nu+1)\ T_{n,\nu}(\alpha)\ ,\ 0\leqslant\nu\leqslant n,$$

with $T_{n,-1}(\alpha)=0$.

# Boas-Buck type sequences

... when there exists a sequence of nonzero numbers $\{\rho_n\}_{n\geqslant 0}$ and $a, b \in \mathbb{R}$ such that for $x \in [a, b]$

$$G(x,t) = A(t)B(xg(t)) = \sum_{n\geqslant 0} P_n(x)\frac{t^n}{\rho_n}$$

where

$$\{A(t), B(t), g(t)\} = \sum_{n\geqslant 0}\{a_n, b_n, g_n\}t^n \text{ satisfying } a_0 \cdot b_n \cdot g_1 \neq 0 \text{ but } g_0 = 0.$$

**Lemma.** Let $g : [a, b] \to \mathbb{R}_+$ with $a, b \in \mathbb{R}$ and $B(x) \in L_1(\mathbb{R}_+; x^{-\gamma}dx)$, with $\gamma > -\alpha$. If $G(x,t) = A(t)B(xg(t))$ is a Boas-Buck GF, then

$$\begin{aligned} KL_\alpha[G(x,t)](\tau) &= \frac{A(t)}{2\pi i\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^2} \\ &\times \int_{1-\gamma-i\infty}^{1-\gamma+i\infty}\Gamma\left(1-s+\alpha+\frac{i\tau}{2}\right)\Gamma\left(1-s+\alpha-\frac{i\tau}{2}\right)B^\star(s)\Big(g(t)\Big)^{-s}ds, \end{aligned} \tag{5}$$

where $\quad B^\star(s) = \int_0^\infty B(x)x^{s-1}dx \quad , \quad \gamma = \Re(s) > -\alpha,$

where the latter integral converges absolutely.

## Boas-Buck type sequences (cont.)

**Theorem.** Let $G(x,t)$ be a Boas-Buck generating function of the MPS $\{P_n\}_{n\geqslant 0}$, then $\{S_n(\cdot) := KL_\alpha[P_n(x)](\cdot)\}_{n\geqslant 0}$ is generated by

$$A(t)KL_\alpha[B(xg(t))](\tau) = \sum_{n\geqslant 0} S_n(\tau)\frac{t^n}{\rho_n}\ ,$$

provided that one of the following conditions is fulfilled :

1. $\forall \tau \in \mathbb{R}_+,\ \ KL_\alpha\left[\left(\frac{\partial^n}{\partial u^n}B(u)\right)\Big|_{u=xt} x^n\right](\tau)$ converges uniformly by $t\in[0,\delta]$, $B(x)\in L_1(\mathbb{R}_+, x^{-\gamma}dx)$ and $(s)_n B^\star(s) \in L_1(1-\gamma-i\infty, 1-\gamma+i\infty)$ for any $n\in\mathbb{N}$ with $\gamma > -\alpha$ and $\delta > 0$.
2. $\sum_{n\geqslant 0}\left|P_n(x)\frac{t^n}{\rho_n}\right| \in L_1(\mathbb{R}_+; x^\alpha K_0(2\sqrt{x})dx)$

# Boas-Buck type sequences (cont.)

**Proposition.** If the MPS $\{P_n\}_{n\geqslant 0}$ is generated

$$A(t)\, G^{m,n}_{p,q}\left(xg(t);\, \begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right) = \sum_{n\geqslant 0} P_n\left(x;\, \begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right) \frac{t^n}{\prod_{\sigma=1}^{n}\rho_\sigma}\ , \tag{6}$$

then the MPS $\{S_n\}_{n\geqslant 0}$ is generated by

$$2A(t)\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^{-2} G^{m+2,n}_{p+2,q}\left(g(t);\, \begin{matrix}-\alpha-i\tau/2, -\alpha+i\tau/2, \mathbf{a}\\ \mathbf{b}\end{matrix}\right)$$

$$= \sum_{n\geqslant 0} S_n\left(\tfrac{\tau^2}{4};\, \begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right) \frac{t^n}{\prod_{\sigma=1}^{n}\rho_\sigma}$$

**Example.** $(1-t)^{-\lambda}\ {}_{p+1}F_q\left(\begin{matrix}\lambda, a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\,\middle|\, -\frac{xt}{1-t}\right) = \sum_{n\geqslant 0} p_n(x;\lambda)\frac{t^n}{n!}$ and

$$(1-t)^{-\lambda}\ {}_{p+3}F_q\left(\begin{matrix}\lambda, a_1,\ldots,a_p, \alpha+1-i\tau/2, \alpha+1+i\tau/2\\ b_1,\ldots,b_q\end{matrix}\,\middle|\, -\frac{t}{1-t}\right)$$

$$= \sum_{n\geqslant 0} q_n(\tau^2/4;\lambda,\alpha)\frac{t^n}{n!}\ .$$

# Parseval identity for $KL_\alpha$

### Theorem

*The operator $KL_\alpha$ is an isomorphism between Hilbert spaces*

$$\begin{array}{rccl} KL_\alpha & : & L_2(\mathbb{R}_+; x^{2\alpha+1}dx) & \rightarrow & L_2\left(\mathbb{R}_+; \tau \sinh(\pi\tau) \left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^4 \frac{d\tau}{4\pi^2}\right) \\ & & f & \mapsto & 2\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^{-2} \int_0^\infty x^\alpha K_{i\tau}(2\sqrt{x}) f(x) dx \end{array}$$

*The following generalized Parseval equality holds*

$$\int_0^\infty x^{2\alpha+1} f(x) g(x) dx$$

$$= \frac{1}{4\pi^2} \int_0^\infty \tau \sinh(\pi\tau) \left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^4 KL_\alpha[f](\tau) KL_\alpha[g](\tau) d\tau \ ,$$

*where $f, g \in L_2(\mathbb{R}_+; x^{2\alpha+1}dx)$.*

**Proposition.** For any $f \in \mathcal{P}$ and any nonnegative function $g \in L_2(\mathbb{R}_+; x^{2\alpha+1}dx)$, we have

$$\int_0^\infty x^{2\alpha+1} f(x) g(x) dx = \frac{1}{4\pi} \int_0^\infty KL_\alpha[f](\tau) \frac{\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^4}{|\Gamma(i\tau)|^2} KL_\alpha[g](\tau) d\tau \ ,$$

as long as $KL_\alpha[g] \in L_1(\mathbb{R}_+ \ , \ \mathrm{e}^{(\frac{\pi}{2}-\delta)\tau}(1+\tau)^{2(\alpha+1)}d\tau)$ for some $\delta \in (0, \pi/2)$

**Corollary.** For any polynomial $f$ and $|\mathrm{Im}\mu| < 2\beta$, it is valid the identity

$$\int_0^\infty x^{\alpha+\beta} f(x) K_{i\mu}(2\sqrt{x})dx$$
$$= \frac{1}{8\pi\Gamma(2\beta)} \int_0^\infty KL_\alpha[f](\tau) \frac{\left|\Gamma\left(\beta + \frac{i(\tau+\mu)}{2}\right)\Gamma\left(\beta + \frac{i(\tau-\mu)}{2}\right)\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^2}{\left|\Gamma(i\tau)\right|^2} d\tau \ .$$

**Corollary.** For any polynomial $f$ and $\beta > 0$, we have

$$\int_0^\infty x^{\alpha+\beta} \mathrm{e}^{-x} f(x)dx$$
$$= \frac{\sqrt{\mathrm{e}}}{4\pi} \int_0^\infty KL_\alpha[f](\tau) \frac{\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\Gamma\left(\beta+\frac{i\tau}{2}\right)\right|^2}{\left|\Gamma(i\tau)\right|^2} W_{-\beta+\frac{1}{2},\frac{i\tau}{2}}(1)d\tau \ , \qquad (7)$$

holds, where $W_{\gamma,\mu}(x)$ represents the Whittaker function.

**Corollary.** For any polynomial $f$ and $|\mathrm{Im}\mu| < 2\beta$, it is valid the identity

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# More properties of $KL_\alpha$

- For any $m, n \in \mathbb{N}_0$ and any $f \in \mathcal{P}$, it is valid

$$KL_\alpha\left[\left(\frac{1}{x}\mathcal{A}x + 2\alpha\frac{d}{dx}x\right)^m x^n f(x)\right](\tau) = (-1)^m\left(\frac{\tau^2}{4}+\alpha^2\right)^m\left|\left(\alpha+1+\frac{i\tau}{2}\right)_n\right|^2 KL_{\alpha+n}[f](\tau).$$

- Let $\{S_n(\cdot;\alpha) := KL_\alpha[P_n](\cdot)\}_{n\geqslant 0}$. If $\{P_n\}_{n\geqslant 0}$ is given by

$$P_n(x) = (-1)^n\frac{\left(\prod_{\nu=1}^{q}(b_\nu)_n\right)}{\left(\prod_{\nu=1}^{p}(a_\nu)_n\right)}\,{}_{p+1}F_q\left(\begin{matrix}-n, a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\,\middle|\, x\right), \quad n \geqslant 0, \tag{8}$$

where the coefficients $a_j, b_k$ with $j = 1, \ldots, p$ and $k = 1, \ldots, q$, do not depend on $x$ but possibly depending on $n$, then

$$S_n\left(\frac{\tau^2}{4}\right) = \frac{(-1)^n\left(\prod_{\nu=1}^{q}(b_\nu)_n\right)}{\left(\prod_{\nu=1}^{p}(a_\nu)_n\right)}\,{}_{p+3}F_q\left(\begin{matrix}-n, a_1, \ldots, a_p, \alpha+1-\frac{i\tau}{2}, \alpha+1+\frac{i\tau}{2}\\ b_1, \ldots, b_q\end{matrix}\,\middle|\, 1\right)$$

## Examples.

**1.** The MPS $\{P_n\}_{n\geqslant 0}$ is said to be an Appell sequence when

$$P'_{n+1}(x) = (n+1)P_n(x)\ ,\ \ n \geqslant 0.$$

If $\{P_n\}_{n\geqslant 0}$ is $d$-orthogonal,

then $\{S_n := KL_\alpha[P_n]\}_{n\geqslant 0}$ is $(2d+2)$-orthogonal and

$$\delta_i S_{n+1}\left(\frac{\tau^2}{4};\alpha\right) = (n+1)S_n\left(\frac{\tau^2}{4};\alpha+\frac{1}{2}\right)\ ,\ \ n\in\mathbb{N}_0,$$

where $\delta_\omega f(x) := \frac{f(x+\omega)-f(x-\omega)}{2\omega}$.

# Examples.

**2.** Let $\{R_n\}_{n\geqslant 0}$ be a reversed Appell sequence, i.e.,

$$R_n(x) = \frac{1}{\lambda_n} x^n P_n\left(\frac{1}{x}\right) \ , \ n \in \mathbb{N}_0,$$

where $\{P_n\}_{n\geqslant 0}$ is an Appell sequence and $\lambda_n = P_n(0) \neq 0$

According to Ben Cheikh & Douak (2001), if $\{R_n\}_{n\geqslant 0}$ is $d$-orthogonal then

$$R_n(x) := R_n(x;\bar{\alpha}_d) = (-1)^n \left(\prod_{\sigma=1}^{d} (\alpha_\sigma + 1)_n\right) {}_1F_d\left(\begin{matrix} -n \\ \alpha_1+1,\ldots,\alpha_d+1 \end{matrix}; x\right) \ , \ n \geqslant 0,$$

The corresponding $KL_\alpha$-transformed sequence $\{S_n\}_{n\geqslant 0}$ with $S_n(\cdot;\alpha,\bar{\alpha}_d) = KL_\alpha[R_n(x;\bar{\alpha}_d)](\cdot)$ is $\widetilde{d}$-orthogonal where

$$\widetilde{d} = \begin{cases} 2 & , \quad d = 1 \\ 1 & , \quad d = 2 \\ d & , \quad d = 3,4,5,\ldots \end{cases} ,$$

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$$\widetilde{d} = \begin{cases} 2 & , \quad d = 1 \\ 1 & , \quad d{=}2 \longrightarrow \text{Continuous Dual Hahn} \\ d & , \quad d = 3, 4, 5, \ldots \end{cases} ,$$

# For instance, ...

- $(d=1)$ : if $u_0 := u_0(\alpha_1)$ is the regular form associated to the Laguerre polynomials $\{\widehat{L}_n(\cdot;\alpha_1)\}_{n\geqslant 0}$, then

$$< u_0(\alpha_1), f >= \int_0^\infty f(x) \frac{e^{-x} x^{\alpha_1}}{\Gamma(\alpha_1+1)} \, dx \ , \ f \in \mathcal{P}.$$

Necessarily, the canonical form $s_0(\alpha_1,\alpha)$ corresponding to the $KL_\alpha$-transform of $\{\widehat{L}_n(\cdot;\alpha_1)\}_{n\geqslant 0}$ admits the representation

$$\langle s_0(\alpha_1,\alpha), g_\alpha \rangle = \frac{\sqrt{e}}{4\pi} \int_0^\infty g_\alpha(\tau) \frac{\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\Gamma\left(\alpha_1-\alpha+\frac{i\tau}{2}\right)\right|^2}{\Gamma(\alpha_1+1)\ \left|\Gamma\left(i\tau\right)\right|^2} W_{\alpha-\alpha_1+\frac{1}{2},\frac{i\tau}{2}}(1)\, d\tau$$

as long as $\alpha_1 > \alpha$, where $g_\alpha(\tau) = KL_\alpha[f](\tau)$.

The recurrence relation can be written as

$$\begin{aligned} S_{n+2}(z;\alpha) &= (z-(-2\alpha+\alpha_1-(\alpha+n)^2))S_{n+1}(z;\alpha) \\ &\quad +2(n+1)(\alpha+n+1)\,(\alpha_1+n+1)\,S_n(z;\alpha) \\ &\quad +n(n+1)\,(\alpha_1+n)\,(\alpha_1+n+1)\,S_{n-1}(z;\alpha)\ , \end{aligned} \tag{9}$$

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# For instance, ...

- $(d=2)$ : if $u_0 := u_0(\alpha_1,\alpha_2)$ is the regular form associated to the 2-orthogonal Laguerre type polynomials $\{\widehat{B}_n(\cdot;\alpha_1,\alpha_2)\}_{n\geqslant 0}$, then

$$< u_0(\alpha_1,\alpha_2), f >= \int_0^\infty f(x)\, \frac{x^{\frac{\alpha_1+\alpha_2}{2}} K_{\alpha_1-\alpha_2}(2\sqrt{x})}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\, dx \ , \ f\in\mathcal{P}.$$

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as long as $\alpha_2>\alpha$, where $g_\alpha(\tau)=KL_\alpha[f](\tau)$.
(Van Assche & Yakubovich, 2000)

From the 3rd order recurrence relation fulfilled by the 2-MOPS $\{\widehat{B}_n(\cdot;\alpha_1,\alpha_2)\}_{n\geqslant 0}$ we deduce that $\{S_n\}_{n\geqslant 0}$ satisfies

$$S_{n+2}(z;\alpha)=\Big(z-\beta_{n+1}\Big)S_{n+1}(z;\alpha)\\ -(n+1)\,(\alpha_1+n+1)\,(\alpha_2+n+1)\,(-2\alpha+\alpha_1+\alpha_2+n)\,S_n(z;\alpha)$$

with

$$\beta_{n+1}=(-\alpha(\alpha+4)+2n^2+(5-2\alpha)n+\alpha_2(2n+3)+\alpha_1\,(\alpha_2+2n+3)+3)$$

In fact, $\{S_n := S_n(\cdot;2;a-1,a+b-1,a+c-1)\}_{n\geqslant 0}$ are precisely the *Continuous Dual Hahn* polynomials.

All the orthogonal polynomial sequences

that are mapped by the $KL_\alpha$-transform

to $d$-orthogonal sequences

# MOPSs whose $KL_\alpha$-transformed sequence is a $d$-MOPS

**Theorem.** Let $\{B_n\}_{n\geqslant 0}$ be a MOPS with respect to $u_0$.

If $\{S_n\}_{n\geqslant 0}$ is a $d$-MOPS, then $d$ is even $\geqslant 2$

and $\{B_n\}_{n\geqslant 0}$ is semiclassical of class $s \in \left\{\max(0, \frac{d}{2}-2)\ ,\ \frac{d}{2}-1\ ,\ \frac{d}{2}\right\}$ insofar as there exist two polynomials $\phi, \psi$ such that $u_0$ fulfills

$$D(\phi u_0) + \psi u_0 = 0\ .$$

Precisely, there is a monic polynomial $\rho$, with $\deg \rho(x) = \frac{d}{2}$, and $N \neq 0$, such that $(\phi, \psi)$ is given by

**a)** $\phi(x) = x^2$ and $\psi(x) = x(N\rho(x) - (3+2\alpha))$ with $\rho(0) = 0$, $\langle u_0, \rho(x)\rangle \neq N^{-1}(2+2\alpha)$ and $\alpha \neq -\frac{n+3}{2}$, $n \in \mathbb{N}_0$ ($u_0$ is of class $s = \frac{d}{2}$);

**b)** $\phi(x) = x$ and $\psi(x) = N\rho(x) - (2+2\alpha)$ with $\langle u_0, \rho(x)\rangle \neq N^{-1}(2+2\alpha)$ and $\alpha \neq -\frac{n}{2} - 1$, for $n \geqslant 1$ ($u_0$ is of class $s = \frac{d}{2} - 1$);

**c)** $\phi(x) = 1$ and $\psi(x) = N\theta_0\rho(x)$ with $N\rho(0) = 1 + 2\alpha$ ($u_0$ is of class $s = \frac{d}{2} - 2$, as long as $d \geqslant 4$).

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# MOPSs whose $KL_\alpha$-transformed sequence is a $d$-MOPS

Moreover, the MOPS $\{B_n\}_{n\geqslant 0}$ fulfills

$$
\begin{aligned}
& x^2 B_n''(x) + x\Big(N\rho(x) - (3+2\alpha)\Big)B_n'(x) \\
& -\Big\{N\rho(x)\Big(N\rho(x) - (2+2\alpha)\Big) - Nx\rho'(x) - x + (1+2\alpha)\Big\}B_n = -\sum_{\nu=n-1}^{n+d} \rho_{n,\nu}^{d} B_\nu(x)
\end{aligned}
$$

where

$$
\rho_{n,\mu}^{d} = \begin{cases} \dfrac{\langle u_0, B_n^2\rangle}{\langle u_0, B_\mu^2\rangle}\alpha_{n+1}^{n+d-\mu} & \text{if} \quad n+1 \leqslant \mu \leqslant n+d, \ \text{with} \ \ n \geqslant 0, \\ \zeta_n - \alpha^2 & \text{if} \quad \mu = n, \ \text{with} \ \ n \geqslant 0, \\ \gamma_1 & \text{if} \quad \mu = n-1 \ \text{with} \ \ n \geqslant 1 \ . \end{cases}
$$

# MOPSs whose $KL_\alpha$-transformed sequence is a 2-MOPS

Two situations arise :

**Case a.** The form $u_0$ is a semiclassical form of class $s = 1$, insofar as it fulfills

$$D(x^2 u_0) + x(Nx - (3 + 2\alpha))u_0 = 0$$

and, therefore the corresponding MOPS $\{B_n\}_{n \geqslant 0}$ can be expressed as

$$B_{n+1}(N^{-1}x) = \widehat{L}_{n+1}(N^{-1}x; 2\alpha + 2) + a_n \widehat{L}_n(N^{-1}x; 2\alpha + 2)$$

$$x\widehat{L}_{n+1}(N^{-1}x; 2\alpha + 2) = B_{n+1}(N^{-1}x) - (a_n - (2n - 2\alpha - 3))B_n(N^{-1}x)$$

where $\{\widehat{L}_n\}_{n \geqslant 0}$ represents the (monic) Laguerre polynomials and

$$a_n = \frac{\lambda(n+1)! + (2\alpha - \lambda + 2)(2\alpha + 3)_{n+1}}{\lambda\, n! + (2\alpha - \lambda + 2)(2\alpha + 3)_n}\ ,\ n \geqslant 0.$$

**Case b.** The MOPS $\{B_n\}_{n \geqslant 0}$ is, up to a linear change of variable, a Laguerre sequence of parameter $\alpha_1$.

# another integral transform

$$T_{\nu,\mu}[f(x)](\tau) := \frac{\Gamma(3/2+\nu-\mu)}{|\Gamma(\nu+1+i\tau)|^2} \int_0^\infty \mathrm{e}^{-x/2} x^{\nu-1/2} W_{\mu,i\tau}(x)\ f(x)\mathrm{d}x$$

$$T_{\nu,\mu}[x^n](\tau) := \frac{1}{(3/2+\nu-\mu)_n}\ |(\nu+1+i\tau)_n|^2$$

For any polynomial $f$, we have

$$T_{\nu,\mu}\left[{}^t\mathcal{A}f\right](\tau) = -(\nu^2+\tau^2)T_{\nu,\mu}[f](\tau)$$

with

$${}^t\mathcal{A} = \frac{d}{dx}x\frac{d}{dx}x - \frac{d}{dx}(x-2\nu)x + x\left(\mu-\nu+\frac{1}{2}\right)$$

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Let $\{P_n\}_{n\geqslant 0}$ and $\{S_n\}_{n\geqslant 0}$ be two polynomial sequences.

If $\{S_n\}_{n\geqslant 0}$ is $d$-orthogonal fulfilling

$$\tau^2 S_n(\tau^2) = S_{n+1}(\tau^2) + \widetilde{\beta}_{n+1} S_n(\tau^2) + \sum_{\sigma=0}^{d} \widetilde{\gamma}_{n-\sigma-1}^{d-\sigma} S_{n-\sigma}(\tau^2)$$

then $\{P_n\}_{n\geqslant 0}$ satisfies

$$\left({}^t\mathcal{A}P_n\right)(x) = -P_{n+1}(x) - (\widetilde{\beta}_{n+1} + \nu^2)P_n(x) - \sum_{\sigma=0}^{d} \widetilde{\gamma}_{n-\sigma-1}^{d-\sigma} P_{n-\sigma}(x)$$

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thank you !