G-function of Meijer and generalized hypergeometric function: interplay of new facts

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Boring but necessary definition

Set $\mathbf{a}=(a_1,a_2,\ldots,a_p)\in\mathbb{C}^p$, $\mathbf{b}=(b_1,b_2,\ldots,b_q)\in\mathbb{C}^q$

Definition of Meijer's G-function

Suppose $0\le m\le q$, $0\le n\le p$ are integers and $\mathbf{a}$, $\mathbf{b}$ are arbitrary complex vectors, such that $a_i-b_j-1\notin\mathbb{N}_0$ for $i=1,\ldots,n$, $j=1,\ldots,m$. Define

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right):=$$
$$\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\frac{\Gamma(b_1+s)\cdots\Gamma(b_m+s)\Gamma(1-a_1-s)\cdots\Gamma(1-a_n-s)z^{-s}}{\Gamma(a_{n+1}+s)\cdots\Gamma(a_p+s)\Gamma(1-b_{m+1}-s)\cdots\Gamma(1-b_q-s)}ds.$$

The contour $\mathcal{L}$ begins and ends at infinity and separates the poles $-b_j-k$, $k=0,1,\ldots$ from the poles $1-a_i+l$, $l=0,1,\ldots$

Properties of G-function I

Here we will mostly deal with $G^{p,0}_{q,p}(z)$.

- The function $G^{m,n}_{p,q}(z)$ is real if the vectors **a**, **b** and the argument z are real. This follows from the fact that all the residues of the integrand are real under these conditions.

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- If $p < q$ then
$$G^{p,0}_{q,p}\left(z\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right) \equiv 0 \quad \text{for } \forall z \in \mathbb{C}$$
and arbitrary values of $\mathbf{a}$ and $\mathbf{b}$.

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  and arbitrary values of **a** and **b**.
- If $p = q$ then

$$G^{p,0}_{p,p}\left(z\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right) = 0 \quad \text{for } |z| > 1$$

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and arbitrary values of **a** and **b**.
- $u(x) = G^{p,0}_{q,p}(\mathbf{b};\mathbf{a};x)$ satisfies the hypergeometric differential equation:
$$\left\{\prod_{j=1}^{p}(D + a_j - 1) - x\prod_{j=1}^{q}(D + b_j)\right\} u(x) = 0, \qquad D := x\frac{d}{dx}.$$

Properties of G-function II

- Introduce the shorthand notation $\mathbf{a}+\alpha=(a_1+\alpha,\ldots,a_p+\alpha)$, $\mathbf{a}_{[k]}=(a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_p)$ and $\Gamma(\mathbf{a})=\prod_{j=1}^{p}\Gamma(a_j)$.

  If the vectors $\mathbf{a}_{[k]}-a_k$ do not contain integers for $k=1,2,\ldots,p$ Meijer's G-function can be expanded as

$$G^{p,0}_{q,p}\left(z\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)$$
$$=\sum_{k=1}^{p} z^{a_k}\frac{\Gamma(\mathbf{a}_{[k]}-a_k)}{\Gamma(\mathbf{b}-a_k)}\,{}_qF_{p-1}\left(\begin{matrix}1-\mathbf{b}+a_k\\ 1-\mathbf{a}_{[k]}+a_k\end{matrix}\,\middle|\,(-1)^{p-q}z\right),$$

  where ${}_qF_{p-1}$ is generalized hypergeometric function (defined on the next slide).

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  where ${}_qF_{p-1}$ is generalized hypergeometric function (defined on the next slide).
- The family of $G^{m,n}_{p,q}$ functions is closed under the reflections $x\to -x$ and $x\to 1/x$, multiplication by powers, fractional differentiation and integration, the Laplace transform, the beta transform and the Mellin (or multiplicative) convolution.

Examples of $G^{p,0}_{p,p}$ functions

Recall

$$ {}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle|\, z\right) = {}_pF_q\left(\mathbf{a};\mathbf{b};z\right) := \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n n!}z^n, $$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is rising factorial.

Examples of $G_{p,p}^{p,0}$ functions

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Write $\psi_p = \sum_{j=1}^{p}(b_j - a_j)$ for the parametric excess. We have:

$$ G_{1,1}^{1,0}\left(z\,\middle|\,\begin{matrix}b_1\\ a_1\end{matrix}\right) = \frac{1}{\Gamma(\psi_1)}z^{a_1}(1-z)^{\psi_1-1} $$

$$ G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}b_1, b_2\\ a_1, a_2\end{matrix}\right) = \frac{z^{a_2}(1-z)^{\psi_2-1}}{\Gamma(\psi_2)}{}_2F_1\left(\begin{matrix}b_1-a_1, b_2-a_1\\ \psi_2\end{matrix};1-z\right) $$

$$ G_{3,3}^{3,0}\left(z\,\middle|\,\begin{matrix}b_1, b_2, b_3\\ a_1, a_2, a_3\end{matrix}\right) = \frac{z^{a_1+a_2-b_1}(1-z)^{\psi_3-1}}{\Gamma(\psi_3)}\times $$

$$ F_3(b_3-a_3, b_1-a_2; b_2-a_3, b_1-a_1; \psi_3; 1-z, 1-1/z), $$

where F_3 is Appell's two-variable hypergeometric function.

The key properties for this presentation 1

- The secret expansion of Niels Erik Nørlund: if $\psi_p \neq 0, -1, -2 \ldots$ then

$$G_{p,p}^{p,0}\left(z \,\middle|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) = \frac{z^{a_k}(1-z)^{\psi_p - 1}}{\Gamma(\psi_p)} \sum_{n=0}^{\infty} \frac{g_p^k(n)}{(\psi_p)_n}(1-z)^n$$

for each $k = 1, 2, \ldots, p$ and different coefficients $g_p^k(n)$ which can be determined from the recurrence

$$g_p^p(n) = \sum_{s=0}^{n} \frac{(b_p - a_k)_{n-s}}{(n-s)!}(\psi_{p-1} + s)g_{s,p-1}^k, \quad k = 1, 2, \ldots, p-1.$$

with initial values $g_1^1(0) = 1$, $g_1^1(n) = 0$, $n \geq 1$. The coefficient $g_p^k(n)$ is obtained by exchanging the roles of a_p and a_k and is independent of a_k and symmetric in the elements of $\mathbf{b}$ and $\mathbf{a}_{[k]}$.

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- Setting $\psi_p = -m$, $m = 0, 1, 2, \dots$ in the above expansion, we get
$$G_{p,p}^{p,0}\left(z \,\middle|\, \begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right) = z^{a_k} \sum_{n=0}^{\infty} \frac{g_p^k(n+m+1)}{n!}(1-z)^n \text{ analytic at z=1!}$$

Generalized Ptolemy's identity

Comparing the free terms in Nørlund's expansion and the residue expansion of $G^{p,0}_{p,p}$ as the sum of ${}_pF_{p-1}$ we get the curious identity:

$$\sum_{k=1}^{p}\Gamma\left[\begin{array}{l}\mathbf{a}_{[k]}-a_k,1-\mathbf{a}_{[k]}+a_k\\ \mathbf{b}-a_k,1-\mathbf{b}+a_k\end{array}\right]=\frac{1}{\Gamma(\psi)\Gamma(1-\psi)}$$

or

$$\sum_{k=1}^{p}\frac{\prod_{j=1}^{p}\sin(\pi(b_j-a_k))}{\prod_{j=1,j\neq k}^{p}\sin(\pi(a_j-a_k))}=\sin\left(\pi\sum\nolimits_{k=1}^{p}(b_k-a_k)\right).$$

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For $p=2$ this is equivalent to Ptolemy's theorem: if a quadrilateral is inscribed in a circle then the product of the lengthes of its diagonals is equal to the sum of the products of the lengthes of the pairs of opposite sides, which can be written as:

$$\sin(\theta_3-\theta_1)\sin(\theta_4-\theta_2)=\sin(\theta_2-\theta_1)\sin(\theta_4-\theta_3)+\sin(\theta_4-\theta_1)\sin(\theta_3-\theta_2)$$

The key properties for this presentation 2

- Well known: if $p > q$ and $\Re(\mathbf{a}) > 0$, then

$$\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})} \int_0^\infty s^{n-1} G_{q,p}^{p,0}\left(s \,\middle|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) ds = \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n}$$

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- Less known: if $p = q$, $\Re(\mathbf{a}) > 0$ and $\Re(\psi) > 0$ then

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- Almost unknown: if $p = q$, $\Re(\mathbf{a}) > 0$, and $\psi = -m$, $m = 0, 1, \ldots$

$$\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int\limits_0^1 s^{n-1} G^{p,0}_{p,p}\left(s\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right) ds$$

$$= \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_p)_n} - \prod_{i=1}^{p}\frac{\Gamma(b_i)}{\Gamma(a_i)}\sum_{j=0}^{m}\frac{g_p^k(j)\Gamma(a_k+n)}{\Gamma(a_k+n+j-m)} \text{ for } n = 0, 1, \ldots$$

Proposition

Suppose: $p \le q+1$, $p = p_1 + p_2$, $q = q_1 + q_2$, where $p_1, q_1, q_2 \ge 0$, $p_2 \ge 1$, $p_2 \ge q_2$. Write $\mathbf{a}_1 = (a_1, \dots, a_{p_1})$, $\mathbf{a}_2 = (a_{p_1+1}, \dots, a_p)$, $\mathbf{b}_1 = (b_1, \dots, b_{q_1})$, $\mathbf{b}_2 = (b_{q_1+1}, \dots, b_q)$ for complex parameter vectors satisfying $\Re(\mathbf{a}_2) > 0$. Then

$$ {}_pF_q(\mathbf{a}_1, \mathbf{a}_2; \mathbf{b}_1, \mathbf{b}_2; -z) = \frac{\Gamma(\mathbf{b}_2)}{\Gamma(\mathbf{a}_2)} \int_0^\infty {}_{p_1}F_{q_1}(\mathbf{a}_1; \mathbf{b}_1; -zt) G^{p_2,0}_{q_2,p_2}\left(t \,\middle|\, \begin{matrix} \mathbf{b}_2 \\ \mathbf{a}_2 \end{matrix}\right) \frac{dt}{t}, $$

where $z \in \mathbb{C}$ if $p_1 \le q_1$ or $z \in \mathbb{C} \backslash (-\infty, -1]$ if $p_1 = q_1 + 1$; If $p_2 = q_2$ the integration is over $[0,1]$ and additional assumption $\Re(\psi_2) > 0$, $\psi_2 = \sum_{i=p_1+1}^{p}(b_i - a_i)$, has to be made. If $p_2 = q_2$ and $\psi_2 = 0$, then

$$ {}_pF_q(\mathbf{a}_1, \mathbf{a}_2; \mathbf{b}_1, \mathbf{b}_2; -z) = \frac{\Gamma(\mathbf{b}_2)}{\Gamma(\mathbf{a}_2)} \left\{ {}_{p_1}F_{q_1}(\mathbf{a}_1; \mathbf{b}_1; -z) + \int_0^1 {}_{p_1}F_{q_1}(\mathbf{a}_1; \mathbf{b}_1; -zt) G^{p_2,0}_{p_2,p_2}\left(t \,\middle|\, \begin{matrix} \mathbf{b}_2 \\ \mathbf{a}_2 \end{matrix}\right) \frac{dt}{t} \right\}, $$

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Particular representations I

Generalized Stieltjes transform representation

$$
{}_{p+1}F_p\left(\begin{matrix}\sigma, \mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\, -z\right) = \frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^1 G^{p,0}_{p,p}\left(t\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\frac{dt}{t(1+zt)^{\sigma}},
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Laplace transform representations:

$$ {}_{p+1}F_p(\mathbf{a};\mathbf{b};-z) = \frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^{\infty} e^{-zt}G^{p+1,0}_{p,p+1}\left(t\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\frac{dt}{t}. $$

$$ {}_{p}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\, -z\right) = \frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^1 e^{-zt}G^{p,0}_{p,p}\left(t\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\frac{dt}{t}. $$

These holds if $\min(\mathbf{a}) > 0$ and also $\psi = \sum(b_i - a_i) > 0$ for the first and the third formula

Particular representations II

If $\psi = 0$, then

$$
{}_pF_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,-z\right) = \frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\left\{e^{-z} + \int_0^1 e^{-zt} G_{p,p}^{p,0}\left(t\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\frac{dt}{t}\right\}.
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If $\Re(\psi) > 1/2$, then

$$ {}_{p-1}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z\right) = \frac{\Gamma(\mathbf{b})}{\sqrt{\pi}\Gamma(\mathbf{b})}\int\limits_0^1 \cos(2\sqrt{zt})\, G^{p,0}_{p,p}\left(t\,\middle|\begin{matrix}\mathbf{b}\\ \mathbf{a}, 1/2\end{matrix}\right)\frac{dt}{t}. $$

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For $\psi = -m$:

$$\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})}{}_3F_2\left(\begin{matrix}\sigma, \mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z\right)$$

$$= \frac{\Gamma(a_2)}{(1+z)^{\sigma}}\sum_{j=0}^{m}\frac{(\mathbf{b}-a_1)_j}{j!\Gamma(a_2+j-m)}{}_2F_1\left(\begin{matrix}\sigma, j-m\\ a_2+j-m\end{matrix}\middle|\frac{z}{1+z}\right)$$

$$+\frac{(\mathbf{b}-a_1)_{m+1}}{(m+1)!}\int_0^1\frac{s^{a_2-1}ds}{(1+sz)^{\sigma}}{}_2F_1\left(\begin{matrix}\mathbf{b}-a_1+m+1\\ m+2\end{matrix}; 1-s\right)$$

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and

$$\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})}\,{}_2F_2\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\, -z\right) = \frac{\Gamma(a_2)}{e^{z}}\sum_{j=0}^{m}\frac{(\mathbf{b}-a_1)_j}{j!\Gamma(a_2+j-m)}\,{}_1F_1\left(\begin{matrix}j-m\\ a_2+j-m\end{matrix}\,\middle|\, z\right)$$

$$+\frac{(\mathbf{b}-a_1)_{m+1}}{(m+1)!}\int_0^1 e^{-zs}s^{a_2-1}\,{}_2F_1\left(\begin{matrix}\mathbf{b}-a_1+m+1\\ m+2\end{matrix};1-s\right)ds.$$

Positivity of G-function I

Observation 1: if $\psi=\sum_{i=1}^{p}(b_i-a_i)>0$ then

$$U(x)=\prod_{i=1}^{p}\frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}=\int\limits_{(0,\infty)}e^{-tx}G_{p,p}^{p,0}\left(e^{-t}\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)dt.$$

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If $\psi=0$ then it follows from Nørlund's secret expansion that

$$\prod_{i=1}^{p}\frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}=\int\limits_{[0,\infty)} e^{-tx}\left\{\delta_0+G_{p,p}^{p,0}\left(e^{-t}\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\right\}dt,$$

where δ_0 is the Dirac measure with mass 1 concentrated at zero.

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Observation 1: if $\psi = \sum_{i=1}^{p}(b_i - a_i) > 0$ then

$$U(x) = \prod_{i=1}^{p} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)} = \int_{(0,\infty)} e^{-tx} G_{p,p}^{p,0}\left(e^{-t} \,\middle|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) dt.$$

If $\psi = 0$ then it follows from Nørlund's secret expansion that

$$\prod_{i=1}^{p} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)} = \int_{[0,\infty)} e^{-tx} \left\{\delta_0 + G_{p,p}^{p,0}\left(e^{-t} \,\middle|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right)\right\} dt,$$

where δ_0 is the Dirac measure with mass 1 concentrated at zero.

Lemma (Alzer, 1997; Grinshpan-Ismail, 2006): the function $U(x)$ is logarithmically completely monotone on $(0,\infty)$ iff

$$v(t) = \sum_{j=1}^{p}(t^{a_j} - t^{b_j}) \geq 0 \quad \text{for } t \in (0,1).$$

Positivity of G-function I

Conclusion: $G^{p,0}_{p,p}\left(t \,\middle|\, \begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right) \geq 0$ on $(0,1)$ if the Müntz polynomial $v(t)=\sum_{j=1}^{p}(t^{a_j}-t^{b_j})\geq 0$ on $(0,1)$. In fact, more is true:

$$p(t)=\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}G^{p,0}_{p,p}\left(e^{-t} \,\middle|\, \begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)$$

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Alzer (1997), follows from Tomić (1949): $U(x)$ is logarithmically completely monotone on $(0,\infty)$ if

$$0\leq a_1\leq a_2\leq\cdots\leq a_p,\quad 0\leq b_1\leq b_2\leq\cdots\leq b_p,$$

$$\text{and}\ \sum_{i=1}^{k}a_i\leq\sum_{i=1}^{k}b_i\ \ \text{for}\ \ k=1,2\ldots,p.$$

These inequalities are known as weak supermajorization and are abbreviated as $\mathbf{b}\prec^{W}\mathbf{a}$, where $\mathbf{a}=(a_1,\ldots,a_p)$, $\mathbf{b}=(b_1,\ldots,b_p)$.

Positivity of G-function II

Observation 2: for $p=2$ $U(x)$ is l. c. m. on $(0,\infty)$ iff $\min(a_1,a_2)\leq\min(b_1,b_2)$ and $a_1+a_2\leq b_1+b_2$ (= Alzer's conditions)

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Grinshpan and Ismail (2006): Suppose that for some $\alpha_j>0$

$$v(t)=\sum_{j=1}^{p}(t^{a_j}-t^{b_j})=\prod_{j=1}^{n}(1-t^{\alpha_j})\geq 0,\quad p=n!/2$$

or for $\alpha_1>\alpha_2>\cdots\alpha_n>0$

$$v(t)=\sum_{j=1}^{p}(t^{a_j}-t^{b_j})=\prod_{1\leq j<k\leq n}(t^{\alpha_k}-t^{\alpha_j})\geq 0,\quad p=2^{n-1}.$$

Then $U(x)$ is l.c.m. on $(0,\infty)$. Grinshpan and Ismail give combinatorial formulas for the numbers a_j, b_j in terms of α_j.

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Observation 3: we can generalize this by taking

$$v(t)=\prod\nolimits_{i=1}^{n}(t^{\beta_i}-t^{\alpha_i}),\quad \text{where } \alpha_i\ge\beta_i\ge 0,\ i=1,\ldots,n.$$

Positivity of G-function III

Theorem 1(K.-Prilepkina, 2014)

Let $I = \{1, 2, \ldots, n\}$ and suppose $\mathcal{I}_{\text{odd}}$ ($\mathcal{I}_{\text{even}}$) comprise all subsets of I having odd (even) number of elements. Suppose that $\alpha_i \ge \beta_i \ge 0$ for $i = 1, \ldots, n$. Then

$$U(x) = \frac{\prod_{J\in\mathcal{I}_{\text{even}}} \Gamma\left(x + \sum_{i\in J}\alpha_i + \sum_{i\in I\setminus J}\beta_i\right)}{\prod_{J\in\mathcal{I}_{\text{odd}}} \Gamma\left(x + \sum_{i\in J}\alpha_i + \sum_{i\in I\setminus J}\beta_i\right)}$$

is logarithmically completely monotone on $(0, \infty)$.

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is logarithmically completely monotone on $(0,\infty)$.

Example: for $\alpha_i\ge\beta_i\ge0$ for $i=1,2,3$ the function U is l.c.m.

$$U(x)=\frac{\Gamma(x+\beta_1+\beta_2+\beta_3)\Gamma(x+\beta_1+\alpha_2+\alpha_3)}{\Gamma(x+\alpha_1+\beta_2+\beta_3)\Gamma(x+\beta_1+\alpha_2+\beta_3)}\times\frac{\Gamma(x+\alpha_1+\beta_2+\alpha_3)\Gamma(x+\alpha_1+\alpha_2+\beta_3)}{\Gamma(x+\beta_1+\beta_2+\alpha_3)\Gamma(x+\alpha_1+\alpha_2+\alpha_3)}$$

Positivity of G-function IV

Observation 4: writing $U'(x)=U(x)(\log U(x))'$ and noting that each function here is a Laplace transform we derive by the convolution theorem the integral equation

$$\log(1/x)G^{p,0}_{p,p}\left(x\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)=\int_x^1 G^{p,0}_{p,p}\left(t\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\sum_{k=1}^{p}\left(\frac{x^{a_k}}{t^{a_k}}-\frac{x^{b_k}}{t^{b_k}}\right)\frac{dt}{t-x}$$

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Observation 5: as we saw

$$v(t)=\sum_{k=1}^{p}(t^{a_k}-t^{b_k})\geq 0 \text{ on } [0,1) \;\Rightarrow\; G_{p,p}^{p,0}\left(x\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\geq 0 \text{ on } (0,1)$$

under additional conditions $a_i\geq 0$, $i=1,\ldots,p$ and $\sum_{i=1}^{p}(b_i-a_i)\geq 0$.

Conjecture: if $a_i\geq 0$, $i=1,\ldots,p$ and $\sum_{i=1}^{p}(b_i-a_i)\geq 0$ then

$$\#\left\{\textbf{zeros of } G_{p,p}^{p,0}\left(x\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right) \textbf{ on } (0,1)\right\}\leq\#\left\{\textbf{zeros of } v(t) \textbf{ on } (0,1)\right\}$$

The limiting cases

Theorem (K.-López, 2014)

Suppose $\mathbf{a}, \mathbf{b} > 0$ and $\psi > 0$. The family of (signed) measures

$$d\rho(s) = \frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})} G^{p,0}_{p,p}\left(s \,\middle|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) ds$$

supported on $[0, 1]$ converges weak-* to the measure

$$\frac{\Gamma(\mathbf{b}^*)}{\Gamma(\mathbf{b}^*)} \left\{ \delta_1 + G^{p,0}_{p,p}\left(s \,\middle|\, \begin{matrix} \mathbf{b}^* \\ \mathbf{a}^* \end{matrix}\right) ds \right\} \quad \text{as } \psi \downarrow 0,$$

where δ_1 denotes the point mass 1 concentrated at $s = 1$, and $(\mathbf{a}^*, \mathbf{b}^*)$ is a point on the hyperplane $\sum(a_i^* - b_i^*) = 0$ such that $\mathbf{a}^* = \lim_{\psi\downarrow 0} \mathbf{a}$, $\mathbf{b}^* = \lim_{\psi\downarrow 0} \mathbf{b}$.
Set $a = \min(a_1, a_2, \ldots, a_p)$. The family $d\rho(s)$ converges weak-* to δ_0 - the point mass 1 concentrated at zero as $a \downarrow 0$.

Applications of positivity I

Luke's inequalities: for all real x

$$e^{f_1 x} \le {}_pF_p(\mathbf{a};\mathbf{b};x) \le 1 - f_1 + f_1 e^x, \quad f_1 = \prod_{i=1}^{p} \frac{a_i}{b_i},$$

for all $x < 1$

$$\frac{1}{(1-f_1x)^{\sigma}} \le {}_{p+1}F_p(\sigma,\mathbf{a};\mathbf{b};x) \le 1 - f_1 + \frac{f_1}{(1-x)^{\sigma}}$$

Luke (1972) stated these inequalities under restrictions $b_i \ge a_i$, $i = 1, 2, \ldots, p$ saying they are "easily proved".

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Theorem (K., 2014)

Luke's inequalities are true if $v(t) \geq 0$. If $x > 0$ conditions can be further weakened and the inequalities can be refined.

Proof: Jensen and converse Jensen inequalities.

Applications of positivity II

Theorem (K.-Sitnik, 2014)

Suppose $\mathbf{a},\mathbf{b}>1$, $0<\sigma\le 1$ and $v(t)\ge 0$. Then for $x>0$

$$ {}_{p+1}F_p(\sigma,\mathbf{a};\mathbf{b};-x)<\frac{1}{\left(1+x\prod_{i=1}^{p}[(a_i-1)/(b_i-1)]\right)^{\sigma}}. $$

Proof: combination of Chebyshev and Jensen plus a hypergeometric identity.

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Proof: combination of Chebyshev and Jensen plus a hypergeometric identity.

Theorem (K.-Prilepkina, 2012)

(a) If $0 < \sigma \leq 1$ and $v(t) \geq 0$, then ${}_{p+1}F_p(\sigma, \mathbf{a}; \mathbf{b}; -x)$ is a Markov function;

(b) If $\sigma > 0$ and $v(t) \geq 0$, then ${}_{p+1}F_p(\sigma, \mathbf{a}; \mathbf{b}; -x)$ is completely monotonic on $(0, \infty)$;

(c) If $v(t) \geq 0$, then ${}_pF_p(\mathbf{a}; \mathbf{b}; -x)$ completely monotonic on $(0, \infty)$.

Applications of positivity III

Theorem (K.-Sitnik, 2014)

Suppose $\delta > 0$ and $v(t) \geq 0$ on $(0,1)$. Then the function

$$x \to \frac{{}_{p+1}F_p\left(\sigma, \mathbf{a}+\delta; \mathbf{b}+\delta; -x\right)}{{}_{p+1}F_p\left(\sigma, \mathbf{a}; \mathbf{b}; -x\right)}$$

is monotone decreasing on $(-1, \infty)$ if $\sigma > 0$ and monotone increasing if $\sigma < 0$.

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Theorem (K.-Sitnik, 2014)

Suppose $v(t) \geq 0$ on $(0,1)$. Then the function

$$\sigma \to {}_{p+1}F_p(\sigma, \mathbf{a}; \mathbf{b}; x)$$

is log-convex on $[0,\infty)$ for each $x < 1$.

What to do if parametric access $\psi < 0$ and/or $\min(\mathbf{a}) < 0$?

What to do if parametric access $\psi < 0$ and/or $\min(\mathbf{a}) < 0$?

Definition. Choose nonnegative integer $n > -\min(\psi, \mathbf{a})$. Define the regularization of the function $G^{p,0}_{p,p}(\mathbf{b}; \mathbf{a}; t)$ as the distribution $\mathcal{G}$ acting according to the formula ($\varphi \in C^{\infty}[0,1]$)

$$\langle \mathcal{G}, \varphi \rangle = \sum_{k=0}^{n-1} \frac{(-1)^k \varphi^{(k)}(1)}{k!} {}_{p+1}F_p \left(\begin{matrix} \mathbf{a}, -k \\ \mathbf{b} \end{matrix} \middle| 1 \right) + \int_0^1 G_n(t) \varphi^{(n)}(t) dt,$$

where

$$G_n(t) = \frac{(-1)^n \Gamma(\mathbf{b})}{\Gamma(\mathbf{a})} G^{p,1}_{p+1,p+1} \left(t \,\middle|\, \begin{matrix} n, \mathbf{b} + n - 1 \\ \mathbf{a} + n - 1, 0 \end{matrix} \right).$$

In particular, $G_0(t) = G^{p,0}_{p,p}(t)$ and the sum is understood to be empty for $n = 0$.

Distributional representations

Theorem (K.-López, 2014)

Suppose $\Re\psi > -n$ and $\Re\mathbf{a} > -n$. Then

$$\langle \mathcal{G}, (1+zt)^{-\sigma} \rangle = {}_{p+1}F_p\left(\begin{matrix}\sigma, \mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z\right)$$

for all $z \in \mathbb{C}\backslash(-\infty, -1]$,

$$\langle \mathcal{G}, e^{-zt} \rangle = {}_pF_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z\right)$$

for all $z \in \mathbb{C}$, and

$$\langle \mathcal{G}, \cos(2\sqrt{zt}) \rangle = {}_{p-1}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z\right)$$

for all $z \in \mathbb{C}$.

Structural conjecture

Conjecture: for any real a and b there exists $N \geq 0$ such that

$$G^{p,1}_{p+1,p+1}\left(t\,\middle|\,\begin{matrix} n, \mathbf{b}+n-1 \\ \mathbf{a}+n-1, 0 \end{matrix}\right) \geq 0$$

for all $n \geq N$.

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for all $n \geq N$.

This would imply for the Gauss type function that

$$\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})}{}_{p+1}F_p\left(\begin{matrix}\sigma, \mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,-z\right) = (1+z)^{-\sigma}R_n(z) + z^n\int_0^1\frac{d\mu_n(t)}{(1+zt)^{\sigma+n}},$$

where

$$R_n(z) = \sum_{k=0}^{n-1}\frac{z^k(\sigma)_k}{k!(1+z)^k}\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})}{}_{p+1}F_p\left(\begin{matrix}\mathbf{a}, -k\\ \mathbf{b}\end{matrix}\,\middle|\,1\right)$$

and

$$d\mu_n(t) = (\sigma)_n G^{p,1}_{p+1,p+1}\left(t\,\middle|\,\begin{matrix}n, \mathbf{b}+n-1\\ \mathbf{a}+n-1, 0\end{matrix}\right)dt.$$

Similarly for the Kummer and Bessel type ...

Equivalent form

Conjecture: for any real a and b there exists $N \geq 0$ such that the sequence

$$f_k = \frac{(-1)^n(\mathbf{a})_{n+k}k!}{(\mathbf{b})_{n+k}(n+k)!} + \frac{1}{(n+k)(n-1)!}\,{}_{p+2}F_{p+1}\left(\begin{matrix}-n-k, -n+1, \mathbf{a}\\ -n-k+1, \mathbf{b}\end{matrix}\middle| 1\right)$$

is a Hausdorff moment sequence for all $n \geq N$

THANK YOU FOR ATTENTION!