Multiple orthogonal polynomials associated with an exponential cubic weight

Walter Van Assche

KU Leuven, Belgium

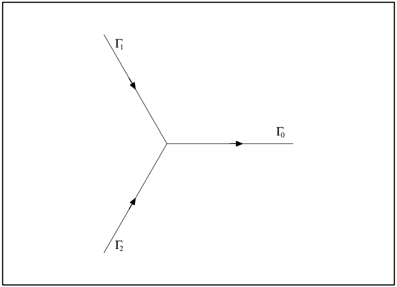
FoCM, Montevideo, December 19, 2014

Joint work with Galina Filipuk and Lun Zhang

Multiple orthogonal polynomials

We are interested in the monic polynomial $P_{n,m}$ of degree $n+m$ for which

$$\int_{\Gamma_0 \cup \Gamma_1} x^j P_{n,m}(x) e^{-x^3}\, dx = 0, \qquad j = 0, 1, \ldots, n-1,$$
$$\int_{\Gamma_0 \cup \Gamma_2} x^j P_{n,m}(x) e^{-x^3}\, dx = 0, \qquad j = 0, 1, \ldots, m-1.$$



Rodrigues formula

Theorem

Let $n, m \in \mathbb{N}$, then

$$e^{-x^3} P_{n,n+m}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3} P_{0,m}(x) \right),$$

$$e^{-x^3} P_{n+m,n}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3} P_{m,0}(x) \right),$$

where $P_{m,0}$ are orthogonal polynomials on $\Gamma_0 \cup \Gamma_1$ and $P_{0,m}$ are orthogonal polynomials on $\Gamma_0 \cup \Gamma_2$.

Rodrigues formula

Theorem

Let $n, m \in \mathbb{N}$, then

$$e^{-x^3} P_{n,n+m}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3} P_{0,m}(x) \right),$$
$$e^{-x^3} P_{n+m,n}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3} P_{m,0}(x) \right),$$

where $P_{m,0}$ are orthogonal polynomials on $\Gamma_0 \cup \Gamma_1$ and $P_{0,m}$ are orthogonal polynomials on $\Gamma_0 \cup \Gamma_2$.

$P_{n,n}$: already considered by Pólya and Szegő (1925).

Rodrigues formula

Theorem

Let $n, m \in \mathbb{N}$, then

$$e^{-x^3} P_{n,n+m}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3} P_{0,m}(x) \right),$$
$$e^{-x^3} P_{n+m,n}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3} P_{m,0}(x) \right),$$

where $P_{m,0}$ are orthogonal polynomials on $\Gamma_0 \cup \Gamma_1$ and $P_{0,m}$ are orthogonal polynomials on $\Gamma_0 \cup \Gamma_2$.

$P_{n,n}$: already considered by Pólya and Szegő (1925).
Special case of Gould-Hopper polynomials (1962).

Orthogonal polynomials, I

Orthogonality relation

$$\int_{\Gamma_0 \cup \Gamma_1} x^k P_{m,0}(x) e^{-x^3} \, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

Orthogonal polynomials, I

Orthogonality relation

$$\int_{\Gamma_0 \cup \Gamma_1} x^k P_{m,0}(x) e^{-x^3}\, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

Recurrence relation

$$xP_{m,0}(x) = P_{m+1,0}(x) + b_m e^{i\pi/3} P_{m,0}(x) + a_m e^{-i\pi/3} P_{m-1,0}(x),$$

where

$$a_n + a_{n+1} = b_n^2,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ and $b_0 = \frac{\Gamma(2/3)}{\Gamma(1/3)}$.

Orthogonal polynomials, II

Orthogonality relation

$$\int_{\Gamma_0 \cup \Gamma_2} x^k P_{0,m}(x) e^{-x^3}\, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

Orthogonal polynomials, II

Orthogonality relation

$$\int_{\Gamma_0 \cup \Gamma_2} x^k P_{0,m}(x) e^{-x^3}\, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

Recurrence relation

$$xP_{0,m}(x) = P_{0,m+1}(x) + b_m e^{-i\pi/3} P_{0,m}(x) + a_m e^{i\pi/3} P_{0,m-1}(x),$$

where

$$a_n + a_{n+1} = b_n^2,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ and $b_0 = \frac{\Gamma(2/3)}{\Gamma(1/3)}$.

Nearest neighbor recurrence relation

$$xP_{n,m}(x) = P_{n+1,m}(x) + c_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x),$$
$$xP_{n,m}(x) = P_{n,m+1}(x) + d_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x).$$

Nearest neighbor recurrence relation

$$xP_{n,m}(x) = P_{n+1,m}(x) + c_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x),$$
$$xP_{n,m}(x) = P_{n,m+1}(x) + d_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x).$$

$$c_{n,n+m} = \begin{cases} b_0 e^{i\pi/3}, & m = 0, \\ -b_{m-1}e^{-i\pi/3}, & m > 0, \end{cases} \qquad d_{n,n+m} = b_m e^{-i\pi/3},$$

$$a_{n,n+m} = \begin{cases} -\frac{n}{3\sqrt{3}}\frac{\Gamma(1/3)}{\Gamma(2/3)}i, \\ -\frac{na_m}{m}e^{i\pi/3}, \end{cases} \qquad b_{n,n+m} = \begin{cases} \frac{n}{3\sqrt{3}}\frac{\Gamma(1/3)}{\Gamma(2/3)}i, & m = 0, \\ -\frac{(n+m)a_m}{m}e^{i\pi/3}, & m > 0. \end{cases}$$

Nearest neighbor recurrence relation

$$xP_{n,m}(x) = P_{n+1,m}(x) + c_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x),$$
$$xP_{n,m}(x) = P_{n,m+1}(x) + d_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x).$$

Nearest neighbor recurrence relation

$$xP_{n,m}(x) = P_{n+1,m}(x) + c_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x),$$
$$xP_{n,m}(x) = P_{n,m+1}(x) + d_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x).$$

$$c_{n+m,n} = b_m e^{i\pi/3}, \qquad d_{n+m,n} = \begin{cases} b_0 e^{-i\pi/3}, & m = 0, \\ -b_{m-1}e^{i\pi/3}, & m > 0, \end{cases}$$

$$a_{n+m,n} = \begin{cases} -\frac{n}{3\sqrt{3}}\frac{\Gamma(1/3)}{\Gamma(2/3)}i, \\ -\frac{(n+m)a_m}{m}e^{-i\pi/3}, \end{cases} \qquad b_{n+m,n} = \begin{cases} \frac{n}{3\sqrt{3}}\frac{\Gamma(1/3)}{\Gamma(2/3)}i, & m = 0, \\ -\frac{na_m}{m}e^{-i\pi/3}, & m > 0. \end{cases}$$

Behavior of a_n, b_n

Theorem (Clarkson, Loureiro, WVA)

There is a unique positive solution of the recurrence relation

$$a_n + a_{n+1} = b_n^2,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ *and* $a_{n+1} > 0$, $b_n > 0$ *for all* $n \in \mathbb{N}$. *This solution corresponds to the initial condition* $b_0 = \frac{\Gamma(2/3)}{\Gamma(1/3)}$.

Behavior of a_n, b_n

Theorem (Clarkson, Loureiro, WVA)

There is a unique positive solution of the recurrence relation

$$a_n + a_{n+1} = b_n^2,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ *and* $a_{n+1} > 0$, $b_n > 0$ *for all* $n \in \mathbb{N}$. *This solution corresponds to the initial condition* $b_0 = \frac{\Gamma(2/3)}{\Gamma(1/3)}$.

The recurrence relation is known as alternative discrete Painlevé I (α-d-P_{I}).

Behavior of a_n, b_n

Theorem (Clarkson, Loureiro, WVA)

There is a unique positive solution of the recurrence relation

$$a_n + a_{n+1} = b_n^2,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ *and* $a_{n+1} > 0$, $b_n > 0$ *for all* $n \in \mathbb{N}$. *This solution corresponds to the initial condition* $b_0 = \frac{\Gamma(2/3)}{\Gamma(1/3)}$.

The recurrence relation is known as alternative discrete Painlevé I (α-d-P_{I}). Eliminating $(a_n)_{n\in\mathbb{N}}$ gives

$$\frac{n}{b_{n-1} + b_n} + \frac{n+1}{b_n + b_{n+1}} = 3b_n^2.$$

Proof of uniqueness

Mapping $T : \mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}_+^{\mathbb{N}}$

$$(Tx)_n = \text{positive solution of } \frac{n+1}{y + x_{n+1}} + \frac{n}{y + x_{n-1}} = 3y^2.$$

Proof of uniqueness

Mapping $T : \mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}_+^{\mathbb{N}}$

$$(Tx)_n = \text{positive solution of } \frac{n+1}{y+x_{n+1}} + \frac{n}{y+x_{n-1}} = 3y^2.$$

- If $x_n \geq 0$ for all $n \geq 0$, then $0 \leq (Tx)_n \leq (2n+1)^{1/3}$, $n \geq 0$.

Proof of uniqueness

Mapping $T : \mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}_+^{\mathbb{N}}$

$$(Tx)_n = \text{positive solution of } \frac{n+1}{y+x_{n+1}} + \frac{n}{y+x_{n-1}} = 3y^2.$$

- If $x_n \geq 0$ for all $n \geq 0$, then $0 \leq (Tx)_n \leq (2n+1)^{1/3}$, $n \geq 0$.
- If $x_n \leq (2n+1)^{1/3}$ for all $n \geq 0$, then $(Tx)_n \geq c_1(2n+1)^{1/3}$ for all $n \geq 1$ and $(Tx)_0 \geq c_0$, with $c_0 = 0.6855$ and $c_1 = 0.638$.

Proof of uniqueness

Mapping $T : \mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}_+^{\mathbb{N}}$

$$(Tx)_n = \text{positive solution of } \frac{n+1}{y+x_{n+1}} + \frac{n}{y+x_{n-1}} = 3y^2.$$

- If $x_n \geq 0$ for all $n \geq 0$, then $0 \leq (Tx)_n \leq (2n+1)^{1/3}$, $n \geq 0$.
- If $x_n \leq (2n+1)^{1/3}$ for all $n \geq 0$, then $(Tx)_n \geq c_1(2n+1)^{1/3}$ for all $n \geq 1$ and $(Tx)_0 \geq c_0$, with $c_0 = 0.6855$ and $c_1 = 0.638$.
- For the norm

$$\|x\| = \sup_{n\geq 0} \frac{|x_n|}{(2n+1)^{1/3}}$$

one has

$$\|Tx - Ty\| \leq 0.9487\|x-y\|.$$

Proof of uniqueness

Mapping $T : \mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}_+^{\mathbb{N}}$

$$(Tx)_n = \text{positive solution of } \frac{n+1}{y+x_{n+1}} + \frac{n}{y+x_{n-1}} = 3y^2.$$

- If $x_n \geq 0$ for all $n \geq 0$, then $0 \leq (Tx)_n \leq (2n+1)^{1/3}$, $n \geq 0$.
- If $x_n \leq (2n+1)^{1/3}$ for all $n \geq 0$, then $(Tx)_n \geq c_1(2n+1)^{1/3}$ for all $n \geq 1$ and $(Tx)_0 \geq c_0$, with $c_0 = 0.6855$ and $c_1 = 0.638$.
- For the norm

$$\|x\| = \sup_{n\geq 0} \frac{|x_n|}{(2n+1)^{1/3}}$$

one has

$$\|Tx - Ty\| \leq 0.9487\|x-y\|.$$

- T is a contraction and hence there exists a unique fixed point: $Tb = b$.

Behavior of a_n, b_n

Theorem

The positive solution of

$$a_n + a_{n+1} = b_n^2,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ *and* $b_0 = \frac{\Gamma(2/3)}{\Gamma(1/3)}$ *has the property that*

$$\lim_{n\to\infty} \frac{a_n}{n^{2/3}} = \frac{1}{2\cdot 3^{2/3}}, \quad \lim_{n\to\infty} \frac{b_n}{n^{1/3}} = \frac{1}{3^{1/3}}.$$

Behavior of a_n, b_n

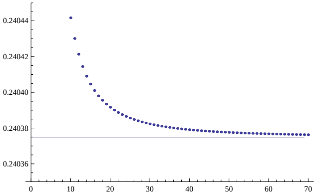
Theorem

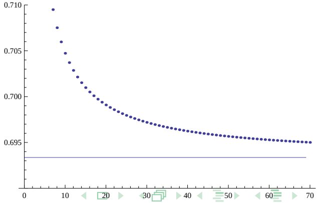
The positive solution of

$$a_n + a_{n+1} = b_n^2,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ *and* $b_0 = \frac{\Gamma(2/3)}{\Gamma(1/3)}$ *has the property that*

$$\lim_{n\to\infty} \frac{a_n}{n^{2/3}} = \frac{1}{2 \cdot 3^{2/3}}, \quad \lim_{n\to\infty} \frac{b_n}{n^{1/3}} = \frac{1}{3^{1/3}}.$$





Zeros

$$P_{n,n}(x) = \frac{(-1)^n}{3^n} e^{x^3} \frac{d^n}{dx^n} e^{-x^3}$$

Zeros

$$P_{n,n}(x) = \frac{(-1)^n}{3^n} e^{x^3} \frac{d^n}{dx^n} e^{-x^3}$$

Theorem

The polynomials $P_{n,n}$ satisfy the symmetry property $P_{n,n}(\omega x) = \omega^{2n} P_{n,n}(x)$, where $\omega = e^{2\pi i/3}$.

Zeros

$$P_{n,n}(x) = \frac{(-1)^n}{3^n} e^{x^3} \frac{d^n}{dx^n} e^{-x^3}$$

Theorem

The polynomials $P_{n,n}$ satisfy the symmetry property $P_{n,n}(\omega x) = \omega^{2n} P_{n,n}(x)$, where $\omega = e^{2\pi i/3}$.
The number of strictly positive real zeros of $P_{n,n}$ is

$$\begin{cases} \frac{2n}{3} & \textit{if } n \equiv 0 \bmod 3, \\ \frac{2(n-1)}{3} & \textit{if } n \equiv 1 \bmod 3, \quad \text{also } 0^2, \\ \frac{2(n-2)}{3} + 1 & \textit{if } n \equiv 2 \bmod 3, \quad \text{also } 0. \end{cases}$$

Zeros

$$P_{n,n}(x) = \frac{(-1)^n}{3^n} e^{x^3} \frac{d^n}{dx^n} e^{-x^3}$$

Theorem

The polynomials $P_{n,n}$ satisfy the symmetry property $P_{n,n}(\omega x) = \omega^{2n} P_{n,n}(x)$, where $\omega = e^{2\pi i/3}$.
The number of strictly positive real zeros of $P_{n,n}$ is

$$\begin{cases} \frac{2n}{3} & \textit{if } n \equiv 0 \bmod 3, \\ \frac{2(n-1)}{3} & \textit{if } n \equiv 1 \bmod 3, \quad \text{also } 0^2, \\ \frac{2(n-2)}{3} + 1 & \textit{if } n \equiv 2 \bmod 3, \quad \text{also } 0. \end{cases}$$

This is the solution of Problem 59 in Part V, Chapter 1 of Aufgaben und Lehrsätze aus der Analysis II (Pólya-Szegő, 1925).

Ratio asymptotics

Theorem

Let K be a compact set in $\mathbb{C} \setminus (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$, then

$$\lim_{n\to\infty} \frac{P_{n,n}(n^{1/3}x)}{P_{n-1,n-1}(n^{1/3}x)} = \Phi(x),$$

uniformly on K, where Φ is the solution of

$$\Phi(x) = \left(x - \frac{1}{3\Phi(x)}\right)^2$$

which behaves as $\Phi(x) = x^2 + \mathcal{O}(1/x)$ as $x \to \infty$.

Ratio asymptotics

Theorem

Let K be a compact set in $\mathbb{C} \setminus (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$, then

$$\lim_{n\to\infty} \frac{P_{n,n}(n^{1/3}x)}{P_{n-1,n-1}(n^{1/3}x)} = \Phi(x),$$

uniformly on K, where Φ is the solution of

$$\Phi(x) = \left(x - \frac{1}{3\Phi(x)}\right)^2$$

which behaves as $\Phi(x) = x^2 + \mathcal{O}(1/x)$ as $x \to \infty$. Furthermore, uniformly on K

$$\lim_{n\to\infty} \frac{1}{n^{2/3}} \frac{P'_{n,n}(n^{1/3}x)}{P_{n,n}(n^{1/3}x)} = 3x^2 - 3\Phi(x).$$

Asymptotic zero distribution

Corollary

Let $\{x_{j,n}, j = 1, \ldots, 2n\}$ *be the zeros of* $P_{n,n}$ *and* μ_n *the normalized counting measure of the scaled zeros* $x_{j,n}/n^{1/3}$. *Then* μ_n *converges weakly to the probability measure* μ *with density*

$$v(x) = \frac{\sqrt{3}}{4\pi}\Big(1 + x[a(x) + b(x)]\Big)\big(b(x) - a(x)\big),$$

where

$$a(x) = \left(\frac{3 - \sqrt{9 - 4x^3}}{2}\right)^{1/3}, \quad b(x) = \left(\frac{3 + \sqrt{9 - 4x^3}}{2}\right)^{1/3}.$$

Asymptotic zero distribution

Corollary

Let $\{x_{j,n}, j = 1, \ldots, 2n\}$ *be the zeros of* $P_{n,n}$ *and* μ_n *the normalized counting measure of the scaled zeros* $x_{j,n}/n^{1/3}$. *Then* μ_n *converges weakly to the probability measure* μ *with density*

$$v(x) = \frac{\sqrt{3}}{4\pi}\Big(1 + x[a(x) + b(x)]\Big)\big(b(x) - a(x)\big),$$

where

$$a(x) = \left(\frac{3 - \sqrt{9 - 4x^3}}{2}\right)^{1/3}, \quad b(x) = \left(\frac{3 + \sqrt{9 - 4x^3}}{2}\right)^{1/3}.$$

This density is related to the Fuss-Catalan density.

Sketch of the proof

$$\frac{1}{N}\frac{\frac{d}{dx}P_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)} = \frac{1}{N^{2/3}}\frac{P'_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)} = \frac{1}{N}\sum_{j=1}^{2n}\frac{1}{x - x_{j,n}/N^{1/3}}.$$

Sketch of the proof

$$\frac{1}{N}\frac{\frac{d}{dx}P_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)} = \frac{1}{N^{2/3}}\frac{P'_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)} = \frac{1}{N}\sum_{j=1}^{2n}\frac{1}{x - x_{j,n}/N^{1/3}}.$$

$$\left|\frac{1}{N^{2/3}}\frac{P'_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)}\right| \leq \frac{2n}{N\delta}, \qquad x \in K \subset \mathbb{C} \setminus (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2).$$

Sketch of the proof

$$\frac{1}{N}\frac{\frac{d}{dx}P_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)} = \frac{1}{N^{2/3}}\frac{P'_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)} = \frac{1}{N}\sum_{j=1}^{2n}\frac{1}{x - x_{j,n}/N^{1/3}}.$$

$$\left|\frac{1}{N^{2/3}}\frac{P'_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)}\right| \leq \frac{2n}{N\delta}, \qquad x \in K \subset \mathbb{C} \setminus (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2).$$

Montel's theorem: there exists a subsequence $(n_k)_k$ such that

$$\lim_{k\to\infty}\frac{1}{n_k^{2/3}}\frac{P'_{n_k-2,n_k-2}(n_k^{1/3}x)}{P_{n_k-2,n_k-2}(n_k^{1/3}x)} = F(x), \qquad x \in K.$$

Sketch of the proof

Structure relation (follows from Rodrigues formula)

$$P_{n-1,n-1}(x) = x^2 P_{n-2,n-2}(x) - \frac{1}{3} P'_{n-2,n-2}(x),$$

implies ratio asymptotics

$$\lim_{k\to\infty} \frac{1}{n_k^{2/3}} \frac{P_{n_k-1,n_k-1}(n_k^{1/3}x)}{P_{n_k-2,n_k-2}(n_k^{1/3}x)} = \Phi(x),$$

where $\Phi(x) = x^2 - \frac{1}{3}F(x)$.

Sketch of the proof

Structure relation (follows from Rodrigues formula)

$$P_{n-1,n-1}(x) = x^2 P_{n-2,n-2}(x) - \frac{1}{3} P'_{n-2,n-2}(x),$$

implies ratio asymptotics

$$\lim_{k\to\infty} \frac{1}{n_k^{2/3}} \frac{P_{n_k-1,n_k-1}(n_k^{1/3}x)}{P_{n_k-2,n_k-2}(n_k^{1/3}x)} = \Phi(x),$$

where $\Phi(x) = x^2 - \frac{1}{3}F(x)$.
Taking derivatives gives

$$\lim_{k\to\infty} \frac{1}{n^{2/3}} \frac{P'_{n_k-1,n_k-1}(n_k^{1/3}x)}{P_{n_k-1,n_k-1}(n_k^{1/3}x)} = \lim_{k\to\infty} \frac{1}{n^{2/3}} \frac{P'_{n_k-2,n_k-2}(n_k^{1/3}x)}{P_{n_k-2,n_k-2}(n_k^{1/3}x)} = F(x).$$

Sketch of the proof

Repeating the same reasoning also gives

$$\lim_{k\to\infty} \frac{1}{n_k^{2/3}} \frac{P_{n_k,n_k}(n_k^{1/3}x)}{P_{n_k-1,n_k-1}(n_k^{1/3}x)} = \Phi(x),$$

Sketch of the proof

Repeating the same reasoning also gives

$$\lim_{k\to\infty} \frac{1}{n_k^{2/3}} \frac{P_{n_k,n_k}(n_k^{1/3}x)}{P_{n_k-1,n_k-1}(n_k^{1/3}x)} = \Phi(x),$$

$$\lim_{k\to\infty} \frac{1}{n^{2/3}} \frac{P'_{n_k,n_k}(n_k^{1/3}x)}{P_{n_k,n_k}(n_k^{1/3}x)} = F(x),$$

Sketch of the proof

Repeating the same reasoning also gives

$$\lim_{k\to\infty} \frac{1}{n_k^{2/3}} \frac{P_{n_k,n_k}(n_k^{1/3}x)}{P_{n_k-1,n_k-1}(n_k^{1/3}x)} = \Phi(x),$$

$$\lim_{k\to\infty} \frac{1}{n^{2/3}} \frac{P'_{n_k,n_k}(n_k^{1/3}x)}{P_{n_k,n_k}(n_k^{1/3}x)} = F(x),$$

$$\lim_{k\to\infty} \frac{1}{n_k^{2/3}} \frac{P_{n_k+1,n_k+1}(n_k^{1/3}x)}{P_{n_k,n_k}(n_k^{1/3}x)} = \Phi(x).$$

Sketch of the proof

Now use the nearest neighbor recurrence relations

$$xP_{n,n}(x) = P_{n+1,n}(x) + c_{n,n}P_{n,n}(x) + a_{n,n}P_{n-1,n}(x) + b_{n,n}P_{n,n-1}(x)$$
$$xP_{n,n}(x) = P_{n,n+1}(x) + d_{n,n}P_{n,n}(x) + a_{n,n}P_{n-1,n}(x) + b_{n,n}P_{n,n-1}(x)$$

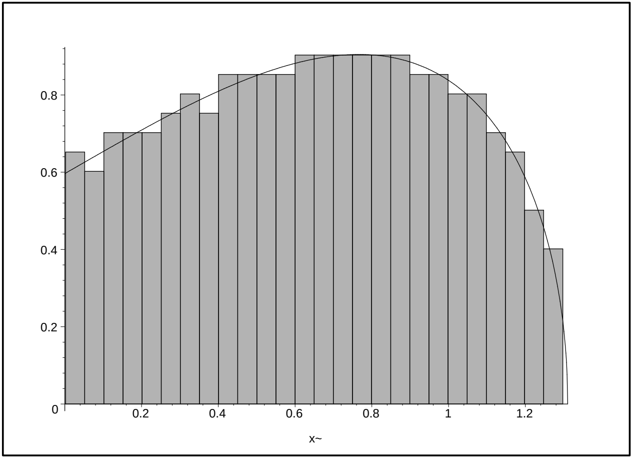
and the relation

$$P_{n+1,n}(x) - P_{n,n+1}(x) = (d_{n,n} - c_{n,n})P_{n,n}(x)$$

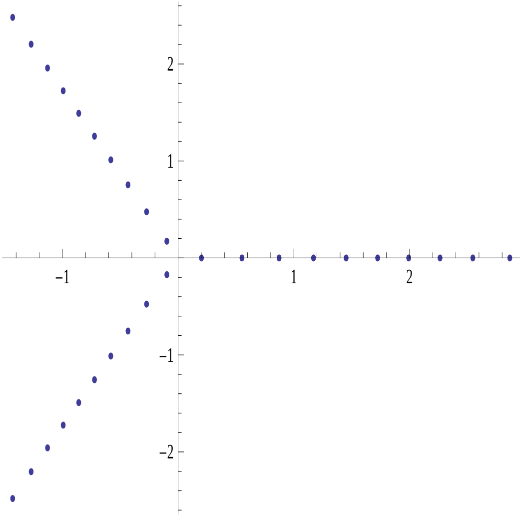
to find

$$\lim_{k\to\infty} \frac{1}{n_k^{1/3}} \frac{P_{n_k+1,n_k}(n_k^{1/3}x)}{P_{n_k,n_k}(n_k^{1/3}x)} = x - \frac{1}{3\Phi(x)}.$$

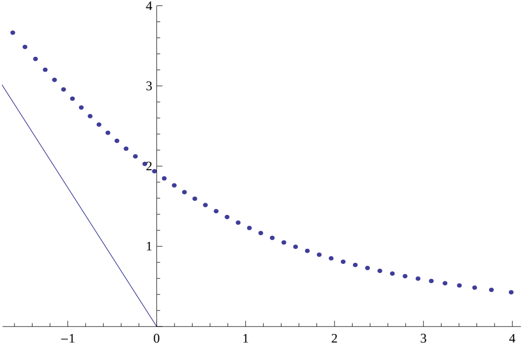
Zero distribution of $P_{600,600}$



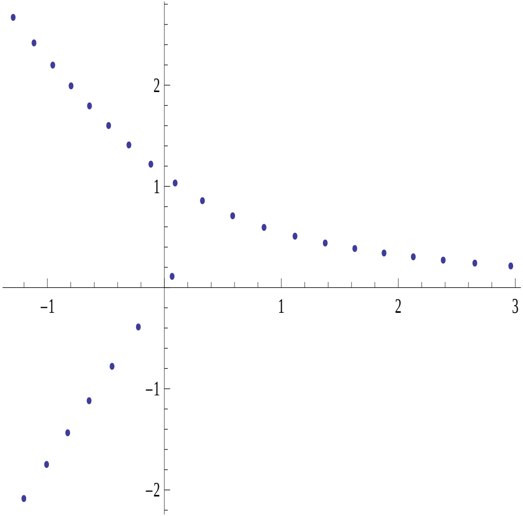
Zeros of $P_{15,15}$



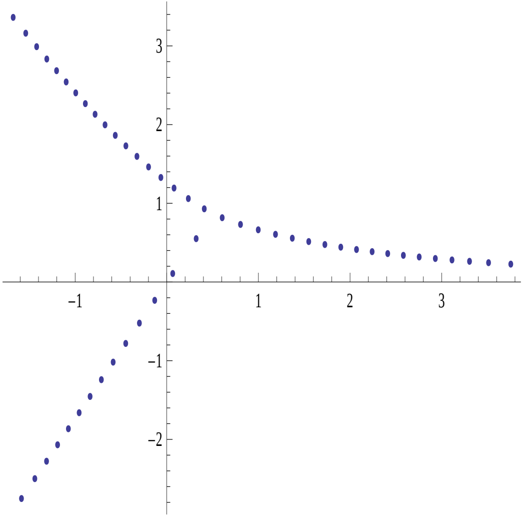
Zeros of $P_{45,0}$



Zeros of $P_{20,7}$



Zeros of $P_{36,14}$



An interesting extension

Monic polynomial $P_{n,m}$ of degree $n+m$ for which

$$\int_{\Gamma_0 \cup \Gamma_1} x^j P_{n,m}(x) e^{-x^3 + tx}\, dx = 0, \qquad j = 0, 1, \ldots, n-1,$$
$$\int_{\Gamma_0 \cup \Gamma_2} x^j P_{n,m}(x) e^{-x^3 + tx}\, dx = 0, \qquad j = 0, 1, \ldots, m-1.$$

An interesting extension

Monic polynomial $P_{n,m}$ of degree $n+m$ for which

$$\int_{\Gamma_0\cup\Gamma_1} x^j P_{n,m}(x) e^{-x^3+tx}\,dx = 0, \qquad j=0,1,\ldots,n-1,$$
$$\int_{\Gamma_0\cup\Gamma_2} x^j P_{n,m}(x) e^{-x^3+tx}\,dx = 0, \qquad j=0,1,\ldots,m-1.$$

Rodrigues formula

$$e^{-x^3+tx} P_{n,n+m}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3+tx} P_{0,m}(x) \right),$$
$$e^{-x^3+tx} P_{n+m,n}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3+tx} P_{m,0}(x) \right),$$

Orthogonal polynomials

Orthogonality relation

$$\int_{\Gamma_0 \cup \Gamma_1} x^k P_{m,0}(x) e^{-x^3+tx}\, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

Orthogonal polynomials

Orthogonality relation

$$\int_{\Gamma_0 \cup \Gamma_1} x^k P_{m,0}(x) e^{-x^3+tx}\, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

Recurrence relation

$$xP_{m,0}(x) = P_{m+1,0}(x) + \hat{b}_m e^{i\pi/3} P_{m,0}(x) + \hat{a}_m e^{-i\pi/3} P_{m-1,0}(x),$$

where

$$\hat{a}_n + \hat{a}_{n+1} = \hat{b}_n^2 - \frac{t}{3},$$

$$3\hat{a}_n(\hat{b}_n + \hat{b}_{n-1}) = n,$$

with $\hat{a}_0 = 0$ and $\hat{b}_0 = \frac{\mathrm{Ai}'(3^{-1/3}t)}{\mathrm{Ai}(3^{-1/3}t)}$.

Behavior of $\hat{a}_n$, $\hat{b}_n$

Theorem (Clarkson, Loureiro, WVA)

For $t \geq 0$ there is a unique positive solution of the recurrence relation

$$a_n + a_{n+1} = b_n^2 - t,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ and $a_{n+1} > 0$, $b_n > 0$ for all $n \in \mathbb{N}$.

Behavior of $\hat{a}_n$, $\hat{b}_n$

Theorem (Clarkson, Loureiro, WVA)

For $t \geq 0$ there is a unique positive solution of the recurrence relation

$$a_n + a_{n+1} = b_n^2 - t,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ and $a_{n+1} > 0$, $b_n > 0$ for all $n \in \mathbb{N}$.

This solution corresponds to a special solution of Painlevé II

$$y''(t) = 2y^3 + ty + n + \frac{1}{2}$$

in terms of Airy functions Ai.

Behavior of $\hat{a}_n$, $\hat{b}_n$

Theorem (Clarkson, Loureiro, WVA)

For $t \geq 0$ there is a unique positive solution of the recurrence relation

$$a_n + a_{n+1} = b_n^2 - t,$$
$$3a_n(b_n + b_{n-1}) = n,$$

with $a_0 = 0$ and $a_{n+1} > 0$, $b_n > 0$ for all $n \in \mathbb{N}$.

This solution corresponds to a special solution of Painlevé II

$$y''(t) = 2y^3 + ty + n + \frac{1}{2}$$

in terms of Airy functions Ai.

$$\frac{n}{\hat{b}_{n-1} + \hat{b}_n} + \frac{n+1}{\hat{b}_n + \hat{b}_{n+1}} = 3\hat{b}_n^2 - t.$$

References

- W. Van Assche, Galina Filipuk, Lun Zhang, *Multiple orthogonal polynomials associated with an exponential cubic weight*, J. Approx. Theory (to appear)
- P. Bleher, A. Deaño, *Topological expansion in the cubic random matrix model*, Int. Math. Res. Notes. (12) (2013), 2699–2755.
- A. Deaño, D. Huybrechs, A.B.J. Kuijlaars, *Asymptotic zero distribution of complex orthogonal polynomials associated with Gaussian quadrature*, J. Approx. Theory **162** (2010), 2202–2224.
- P.A. Clarkson, A.F. Loureiro, W. Van Assche, *Unique positive solution for alternative* d-P_I, in preparation.