

Semi-classical orthogonal polynomials and the Painlevé equations

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Outline

1. Introduction
2. Orthogonal polynomials
3. Some properties of the **fourth Painlevé equation**

$$\frac{d^2q}{dz^2} = \frac{1}{2q} \left(\frac{dq}{dz} \right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q} \quad P_{\text{IV}}$$

4. Fourth Painlevé equation and semi-classical orthogonal polynomials

$$\begin{aligned}\omega(x; t) &= x^\nu \exp(-x^2 + tx), & x \in \mathbb{R}^+, \quad t \in \mathbb{R}, \quad \nu > -1 \\ \omega(x; t) &= |x|^\nu \exp(-x^2 + tx), & x, t \in \mathbb{R}, \quad \nu > -1\end{aligned}$$

5. Third Painlevé equation and semi-classical orthogonal polynomials
6. Fifth Painlevé equation and semi-classical orthogonal polynomials
7. Orthogonal polynomials with discontinuous weights
8. Discrete orthogonal polynomials
9. Conclusions

References

- **P A Clarkson & K Jordaan**, “The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation”, *Constr. Approx.*, **39** (2014) 223–254 [Special Issue: *Painlevé Equations – Part I*; Editors: Percy Deift & Alexander Its]
- **P A Clarkson**, “Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations”, *J. Phys. A* **46** (2013) 185205
- **J G Smith & P A Clarkson**, “The fifth Painlevé equation and semi-classical orthogonal polynomials”, in preparation.
- **J G Smith & P A Clarkson**, “The fifth Painlevé equation and orthogonal polynomials associated with discontinuous weights”, in preparation.

Painlevé Equations

$\frac{d^2q}{dz^2} = 6q^2 + z$	P_I
$\frac{d^2q}{dz^2} = 2q^3 + zq + A$	P_{II}
$\frac{d^2q}{dz^2} = \frac{1}{q} \left(\frac{dq}{dz} \right)^2 - \frac{1}{z} \frac{dq}{dz} + \frac{Aq^2 + B}{z} + Cq^3 + \frac{D}{q}$	P_{III}
$\frac{d^2q}{dz^2} = \frac{1}{2q} \left(\frac{dq}{dz} \right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q}$	P_{IV}
$\begin{aligned} \frac{d^2q}{dz^2} = & \left(\frac{1}{2q} + \frac{1}{q-1} \right) \left(\frac{dq}{dz} \right)^2 - \frac{1}{z} \frac{dq}{dz} + \frac{(q-1)^2}{z^2} \left(Aq + \frac{B}{q} \right) \\ & + \frac{Cq}{z} + \frac{Dq(q+1)}{q-1} \end{aligned}$	P_V
$\begin{aligned} \frac{d^2q}{dz^2} = & \frac{1}{2} \left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-z} \right) \left(\frac{dq}{dz} \right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{q-z} \right) \frac{dq}{dz} \\ & + \frac{q(q-1)(q-z)}{z^2(z-1)^2} \left\{ A + \frac{Bz}{q^2} + \frac{C(z-1)}{(q-1)^2} + \frac{Dz(z-1)}{(q-z)^2} \right\} \end{aligned}$	P_{VI}

with A, B, C and D arbitrary constants.

Painlevé σ -Equations

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2z\frac{d\sigma}{dz} - 2\sigma = 0 \quad S_I$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2\frac{d\sigma}{dz}\left(z\frac{d\sigma}{dz} - \sigma\right) = \frac{1}{4}\beta^2 \quad S_{II}$$

$$\left(z\frac{d^2\sigma}{dz^2} - \frac{d\sigma}{dz}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^2\left(z\frac{d\sigma}{dz} - 2\sigma\right) + 4z\vartheta_\infty\frac{d\sigma}{dz} = z^2\left(z\frac{d\sigma}{dz} - 2\sigma + 2\vartheta_0\right) \quad S_{III}$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 - 4\left(z\frac{d\sigma}{dz} - \sigma\right)^2 + 4\frac{d\sigma}{dz}\left(\frac{d\sigma}{dz} + 2\vartheta_0\right)\left(\frac{d\sigma}{dz} + 2\vartheta_\infty\right) = 0 \quad S_{IV}$$

$$\left(z\frac{d^2\sigma}{dz^2}\right)^2 - \left[2\left(\frac{d\sigma}{dz}\right)^2 - z\frac{d\sigma}{dz} + \sigma\right]^2 + 4\prod_{j=1}^4\left(\frac{d\sigma}{dz} + \kappa_j\right) = 0 \quad S_V$$

$$\frac{d\sigma}{dz}\left[z(z-1)\frac{d^2\sigma}{dz^2}\right]^2 + \left[\frac{d\sigma}{dz}\left\{2\sigma - (2z-1)\frac{d\sigma}{dz}\right\} + \kappa_1\kappa_2\kappa_3\kappa_4\right]^2 = \prod_{j=1}^4\left(\frac{d\sigma}{dz} + \kappa_j^2\right) \quad S_{VI}$$

where $\beta, \vartheta_0, \vartheta_\infty$ and $\kappa_1, \dots, \kappa_4$ are arbitrary constants.

Some History

- The relationship between semi-classical orthogonal polynomials and integrable equations dates back to **Shohat [1939]**, **Freud [1976]**, **Bonan & Nevai [1984]**.
- Subsequently **Fokas, Its & Kitaev [1991, 1992]** identified these integrable equations as **discrete Painlevé equations**.
- **Magnus [1995]** considered the **Freud weight**

$$\omega(x; t) = \exp\left(-\frac{1}{4}x^4 - tx^2\right), \quad x, t \in \mathbb{R},$$

and showed that the coefficients in the three-term recurrence relation can be expressed in terms of solutions of

$$q_n(q_{n-1} + q_n + q_{n+1}) + 2tq_n = n$$

which is discrete P_I (dP_I), as shown by **Bonan & Nevai [1984]**, and

$$\frac{d^2q_n}{dt^2} = \frac{1}{2q_n} \left(\frac{dq_n}{dt} \right)^2 + \frac{3}{2}q_n^3 + 4tq_n^2 + 2(t^2 + \frac{1}{2}n)q_n - \frac{n^2}{2q_n}$$

which is P_{IV} with $A = -\frac{1}{2}n$ and $B = -\frac{1}{2}n^2$.

- **Filipuk, van Assche & Zhang [2012]** commented

“We note that for classical orthogonal polynomials (Hermite, Laguerre, Jacobi) one knows these recurrence coefficients explicitly in contrast to non-classical weights”.

Orthogonal Polynomials

- Monic orthogonal polynomials
- Semi-classical orthogonal polynomials

Monic Orthogonal Polynomials

Let $P_n(x)$, $n = 0, 1, 2, \dots$, be the **monic orthogonal polynomials** of degree n in x , with respect to the positive weight $\omega(x)$, such that

$$\int_a^b P_m(x)P_n(x)\omega(x)dx = h_n\delta_{m,n}, \quad h_n > 0, \quad m, n = 0, 1, 2, \dots$$

One of the important properties that orthogonal polynomials have is that they satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

where the recurrence coefficients are given by

$$\alpha_n = \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\tilde{\Delta}_n}{\Delta_n}, \quad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}$$

with

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix}, \quad \tilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}$$

and $\mu_k = \int_a^b x^k \omega(x)dx$ are the **moments** of the weight $\omega(x)$.

Further Properties

- The Hankel determinant

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix}, \quad \mu_k = \int_a^b x^k \omega(x) dx$$

also has the integral representation

$$\Delta_n = \frac{1}{n!} \int_a^b \cdots \int_a^b \prod_{\ell=1}^n \omega(x_\ell) \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 dx_1 \dots dx_n, \quad n \geq 1$$

which is the **partition function** in random matrix theory.

- The monic polynomials $P_n(x)$ can be uniquely expressed as

$$P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}$$

- The normalization constants can be expressed as

$$h_n = \frac{\Delta_{n+1}}{\Delta_n}, \quad h_0 = \Delta_1 = \mu_0$$

Semi-classical Orthogonal Polynomials

Consider the **Pearson equation** satisfied by the weight $\omega(x)$

$$\frac{d}{dx}[\sigma(x)\omega(x)] = \tau(x)\omega(x)$$

- **Classical orthogonal polynomials:** $\sigma(x)$ and $\tau(x)$ are polynomials with $\deg(\sigma) \leq 2$ and $\deg(\tau) = 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
Hermite	$\exp(-x^2)$	1	$-2x$
Associated Laguerre	$x^\nu \exp(-x)$	x	$1 + \nu - x$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$\beta - \alpha - (2 + \alpha + \beta)x$

- **Semi-classical orthogonal polynomials:** $\sigma(x)$ and $\tau(x)$ are polynomials with either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
Freud	$\exp(-\frac{1}{4}x^4 - tx^2)$	1	$-2tx - x^3$
semi-classical Laguerre	$x^\nu \exp(-x^2 + tx)$	x	$1 + \nu + tx - 2x^2$

If the weight has the form

$$\omega(x; t) = \omega_0(x) \exp(tx)$$

where $\omega_0(x)$ is a classical weight with finite moments and $\int_{-\infty}^{\infty} x^k \omega_0(x) \exp(tx) dx$ exists for all $k \geq 0$. Then:

- the recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the **Toda system**

$$\frac{d\alpha_n}{dt} = \beta_{n+1} - \beta_n, \quad \frac{d\beta_n}{dt} = \beta_n(\alpha_n - \alpha_{n-1})$$

- the k th moment is given by

$$\mu_k(t) = \int_{-\infty}^{\infty} x^k \omega_0(x) \exp(tx) dx = \frac{d^k}{dt^k} \left(\int_{-\infty}^{\infty} \omega_0(x) \exp(tx) dx \right) = \frac{d^k \mu_0}{dt^k}$$

- Since $\mu_k(t) = \frac{d^k \mu_0}{dt^k}$, then $\Delta_n(t)$ and $\tilde{\Delta}_n(t)$ can be expressed as Wronskians

$$\Delta_n(t) = \begin{vmatrix} \mu_0(t) & \mu_1(t) & \dots & \mu_{n-1}(t) \\ \mu_1(t) & \mu_2(t) & \dots & \mu_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(t) & \mu_n(t) & \dots & \mu_{2n-2}(t) \end{vmatrix} = \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}}\right)$$

$$\begin{aligned} \tilde{\Delta}_n(t) &= \begin{vmatrix} \mu_0(t) & \mu_1(t) & \dots & \mu_{n-2}(t) & \mu_n(t) \\ \mu_1(t) & \mu_2(t) & \dots & \mu_{n-1}(t) & \mu_{n+1}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1}(t) & \mu_n(t) & \dots & \mu_{2n-3}(t) & \mu_{2n-1}(t) \end{vmatrix} \\ &= \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-2}\mu_0}{dt^{n-2}}, \frac{d^n\mu_0}{dt^n}\right) = \frac{d}{dt} \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}}\right) \end{aligned}$$

$$\Rightarrow \boxed{\frac{\tilde{\Delta}_n(t)}{\Delta_n(t)} = \frac{d}{dt} \ln \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}}\right)}$$

- the Hankel determinant $\Delta_n(t)$ satisfies the **Toda equation**

$$\frac{d^2}{dt^2} \ln \Delta_n(t) = \frac{\Delta_{n-1}(t)\Delta_{n+1}(t)}{\Delta_n^2(t)}$$

Some Properties of the Fourth Painlevé Equation and the Fourth Painlevé σ -Equation

$$\frac{d^2q}{dz^2} = \frac{1}{2q} \left(\frac{dq}{dz} \right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q} \quad P_{\text{IV}}$$

$$\left(\frac{d^2\sigma}{dz^2} \right)^2 - 4 \left(z \frac{d\sigma}{dz} - \sigma \right)^2 + 4 \frac{d\sigma}{dz} \left(\frac{d\sigma}{dz} + 2\vartheta_0 \right) \left(\frac{d\sigma}{dz} + 2\vartheta_\infty \right) = 0 \quad S_{\text{IV}}$$

- **Hamiltonian Representation**
- **Parabolic Cylinder Function Solutions**

Hamiltonian Representation of P_{IV}

P_{IV} can be written as the **Hamiltonian system**

$$\begin{aligned}\frac{dq}{dz} &= \frac{\partial \mathcal{H}_{\text{IV}}}{\partial p} = 4qp - q^2 - 2zq - 2\vartheta_0 \\ \frac{dp}{dz} &= -\frac{\partial \mathcal{H}_{\text{IV}}}{\partial q} = -2p^2 + 2pq + 2zp - \vartheta_\infty\end{aligned}$$

where $\mathcal{H}_{\text{IV}}(q, p, z; \vartheta_0, \vartheta_\infty)$ is the Hamiltonian defined by

$$\mathcal{H}_{\text{IV}}(q, p, z; \vartheta_0, \vartheta_\infty) = 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

Eliminating p then q satisfies

$$\frac{d^2q}{dz^2} = \frac{1}{2q} \left(\frac{dq}{dz} \right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 + \vartheta_0 - 2\vartheta_\infty - 1)q - \frac{2\vartheta_0^2}{q}$$

which is P_{IV} with $A = 1 - \vartheta_0 + 2\vartheta_\infty$ and $B = -2\vartheta_0^2$, whilst eliminating q then p satisfies

$$\frac{d^2p}{dz^2} = \frac{1}{2p} \left(\frac{dq}{dz} \right)^2 + 6p^3 - 8zp^2 + 2(z^2 - 2\vartheta_0 + \vartheta_\infty + 1)p - \frac{\vartheta_\infty^2}{2p}$$

and letting $p = -\frac{1}{2}q$ gives P_{IV} with $A = 2\vartheta_0 - \vartheta_\infty - 1$ and $B = -2\vartheta_\infty^2$.

Theorem

(Okamoto [1986])

The function

$$\sigma(z; \vartheta_0, \vartheta_\infty) = \mathcal{H}_{\text{IV}} \equiv 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

where q and p satisfy the Hamiltonian system

$$\frac{dq}{dz} = 4qp - q^2 - 2zq - 2\vartheta_0, \quad \frac{dp}{dz} = -2p^2 + 2pq + 2zp - \vartheta_\infty \quad \mathbf{H}_{\text{IV}}$$

satisfies the second-order, second-degree equation

$$\left(\frac{d^2\sigma}{dz^2} \right)^2 - 4 \left(z \frac{d\sigma}{dz} - \sigma \right)^2 + 4 \frac{d\sigma}{dz} \left(\frac{d\sigma}{dz} + 2\vartheta_0 \right) \left(\frac{d\sigma}{dz} + 2\vartheta_\infty \right) = 0 \quad \mathbf{S}_{\text{IV}}$$

Conversely, if $\sigma(z; \vartheta_0, \vartheta_\infty)$ is a solution of \mathbf{S}_{IV} , then

$$q(z; \vartheta_0, \vartheta_\infty) = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\vartheta_\infty)}, \quad p(z; \vartheta_0, \vartheta_\infty) = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\vartheta_0)}$$

are solutions of the Hamiltonian system \mathbf{H}_{IV} .

Proof. If

$$\sigma(z; \vartheta_0, \vartheta_\infty) = 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q \quad (2)$$

where q and p satisfy

$$\frac{dq}{dz} = 4qp - q^2 - 2zq - 2\vartheta_0, \quad \frac{dp}{dz} = -2p^2 + 2pq + 2zp - \vartheta_\infty$$

then differentiating (2) yields

$$\frac{d\sigma}{dz} = -2qp, \quad \frac{d^2\sigma}{dz^2} = -4qp^2 - 2q^2p + 4\theta_0p + 2\theta_\infty q$$

and so

$$\begin{aligned} \frac{d^2\sigma}{dz^2} - 2z\frac{d\sigma}{dz} + 2\sigma &= 2\left(\frac{d\sigma}{dz} + 2\theta_\infty\right)q \\ \frac{d^2\sigma}{dz^2} + 2z\frac{d\sigma}{dz} - 2\sigma &= 4\left(\frac{d\sigma}{dz} + 2\theta_0\right)p \\ \Rightarrow \quad q &= \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\theta_\infty)}, \quad p = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\theta_0)} \end{aligned}$$

Substituting these into (2) yields

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 - 4\left(z\frac{d\sigma}{dz} - \sigma\right)^2 + 4\frac{d\sigma}{dz}\left(\frac{d\sigma}{dz} + 2\vartheta_0\right)\left(\frac{d\sigma}{dz} + 2\vartheta_\infty\right) = 0$$

Parabolic Cylinder Function Solutions of P_{IV}

Theorem

Suppose $\tau_{\nu,n}(z; \varepsilon)$ is given by

$$\tau_{\nu,n}(z; \varepsilon) = \mathcal{W} \left(\varphi_\nu(z; \varepsilon), \varphi'_\nu(z; \varepsilon), \dots, \varphi_\nu^{(n-1)}(z; \varepsilon) \right), \quad n \geq 1$$

where $\tau_{\nu,0}(z; \varepsilon) = 1$ and $\varphi_\nu(z; \varepsilon)$ satisfies

$$\frac{d^2\varphi_\nu}{dz^2} - 2\varepsilon z \frac{d\varphi_\nu}{dz} + 2\varepsilon\nu\varphi_\nu = 0, \quad \varepsilon^2 = 1$$

Then solutions of P_{IV}

$$\frac{d^2q}{dz^2} = \frac{1}{2q} \left(\frac{dq}{dz} \right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q}$$

are given by

$$q_{\nu,n}^{[1]}(z) = -2z + \varepsilon \frac{d}{dz} \ln \frac{\tau_{\nu,n+1}(z; \varepsilon)}{\tau_{\nu,n}(z; \varepsilon)}, \quad (A_1, B_1) = (\varepsilon(2n - \nu), -2(\nu + 1)^2)$$

$$q_{\nu,n}^{[2]}(z) = \varepsilon \frac{d}{dz} \ln \frac{\tau_{\nu,n+1}(z; \varepsilon)}{\tau_{\nu+1,n}(z; \varepsilon)}, \quad (A_2, B_2) = (-\varepsilon(n + \nu), -2(\nu - n + 1)^2)$$

$$q_{\nu,n}^{[3]}(z) = -\varepsilon \frac{d}{dz} \ln \frac{\tau_{\nu+1,n}(z; \varepsilon)}{\tau_{\nu,n}(z; \varepsilon)}, \quad (A_3, B_3) = (\varepsilon(2\nu - n + 1), -2n^2)$$

Parabolic Cylinder Function Solutions of S_{IV}

Theorem

Suppose $\tau_{\nu,n}(z; \varepsilon)$ is given by

$$\tau_{\nu,n}(z; \varepsilon) = \mathcal{W} \left(\varphi_\nu(z; \varepsilon), \varphi'_\nu(z; \varepsilon), \dots, \varphi_\nu^{(n-1)}(z; \varepsilon) \right), \quad n \geq 1$$

where $\tau_{\nu,0}(z; \varepsilon) = 1$ and $\varphi_\nu(z; \varepsilon)$ satisfies

$$\frac{d^2\varphi_\nu}{dz^2} - 2\varepsilon z \frac{d\varphi_\nu}{dz} + 2\varepsilon\nu\varphi_\nu = 0, \quad \varepsilon^2 = 1$$

Then solutions of S_{IV}

$$\left(\frac{d^2\sigma}{dz^2} \right)^2 - 4 \left(z \frac{d\sigma}{dz} - \sigma \right)^2 + 4 \frac{d\sigma}{dz} \left(\frac{d\sigma}{dz} + 2\vartheta_0 \right) \left(\frac{d\sigma}{dz} + 2\vartheta_\infty \right) = 0$$

are given by

$$\sigma_{\nu,n}(z) = \frac{d}{dz} \ln \tau_{\nu,n}(z; \varepsilon), \quad (\vartheta_0, \vartheta_\infty) = (\varepsilon(\nu - n + 1), -\varepsilon n)$$

$$\frac{d^2\varphi_\nu}{dz^2} - 2\varepsilon z \frac{d\varphi_\nu}{dz} + 2\varepsilon\nu\varphi_\nu = 0, \quad \varepsilon^2 = 1 \quad (*)$$

- If $\nu \notin \mathbb{Z}$

$$\varphi_\nu(z; \varepsilon) = \begin{cases} \{C_1 D_\nu(\sqrt{2}z) + C_2 D_\nu(-\sqrt{2}z)\} \exp\left(\frac{1}{2}z^2\right), & \text{if } \varepsilon = 1 \\ \{C_1 D_{-\nu-1}(\sqrt{2}z) + C_2 D_{-\nu-1}(-\sqrt{2}z)\} \exp\left(-\frac{1}{2}z^2\right), & \text{if } \varepsilon = -1 \end{cases}$$

- If $\nu = n \in \mathbb{Z}$, with $n \geq 0$

$$\varphi_n(z; \varepsilon) = \begin{cases} C_1 H_n(z) + C_2 \exp(z^2) \frac{d^n}{dz^n} \{\operatorname{erfi}(z) \exp(-z^2)\}, & \text{if } \varepsilon = 1 \\ C_1 H_n(iz) + C_2 \exp(-z^2) \frac{d^n}{dz^n} \{\operatorname{erfc}(z) \exp(z^2)\}, & \text{if } \varepsilon = -1 \end{cases}$$

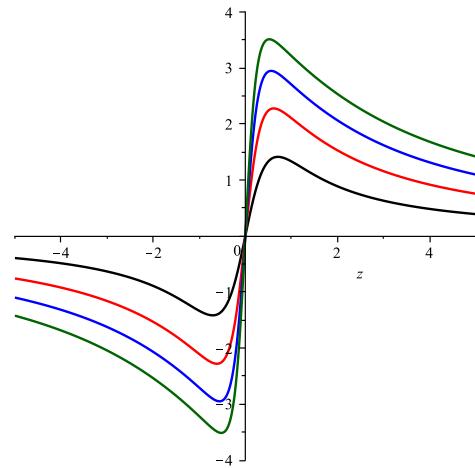
- If $\nu = -n \in \mathbb{Z}$, with $n \geq 1$

$$\varphi_{-n}(z; \varepsilon) = \begin{cases} C_1 H_{n-1}(iz) \exp(z^2) + C_2 \frac{d^{n-1}}{dz^{n-1}} \{\operatorname{erfc}(z) \exp(z^2)\}, & \text{if } \varepsilon = 1 \\ C_1 H_{n-1}(z) \exp(-z^2) + C_2 \frac{d^{n-1}}{dz^{n-1}} \{\operatorname{erfi}(z) \exp(-z^2)\}, & \text{if } \varepsilon = -1 \end{cases}$$

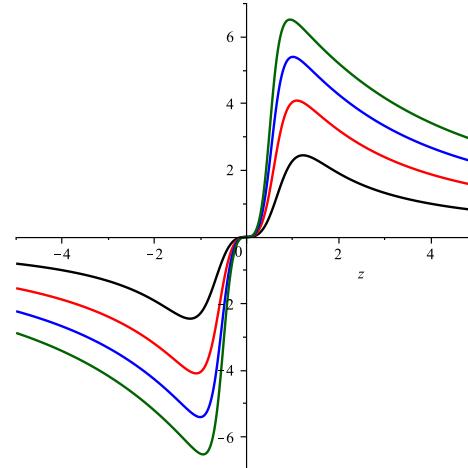
with C_1 and C_2 arbitrary constants, $D_\nu(\zeta)$ the **parabolic cylinder function**, $H_n(z)$ the **Hermite polynomial**, $\operatorname{erfc}(z)$ the **complementary error function** and $\operatorname{erfi}(z)$ the **imaginary error function**.

Plots of Bounded Rational Solutions of S_{IV}

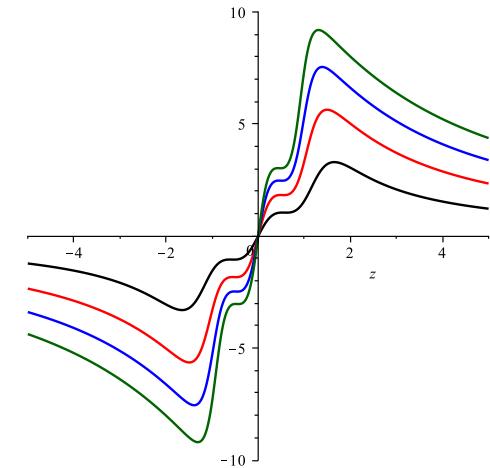
$$\sigma_{m,n}(z) = \frac{d}{dz} \ln \mathcal{W}(H_m(z), H_{m+1}(z), \dots, H_{m+n-1}(z))$$



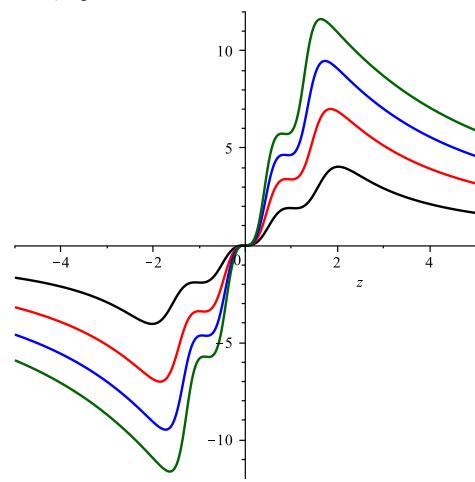
$\sigma_{1,2j}(z), \ j = 1, 2, 3, 4$



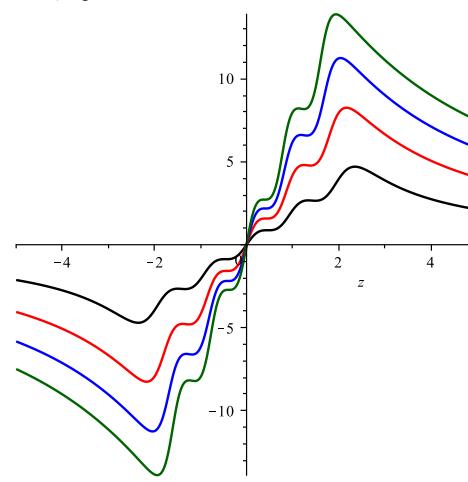
$\sigma_{2,2j}(z), \ j = 1, 2, 3, 4$



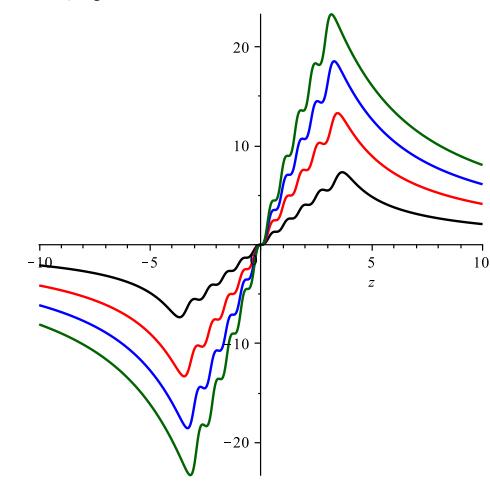
$\sigma_{3,2j}(z), \ j = 1, 2, 3, 4$



$\sigma_{4,2j}(z), \ j = 1, 2, 3, 4$



$\sigma_{5,2j}(z), \ j = 1, 2, 3, 4$

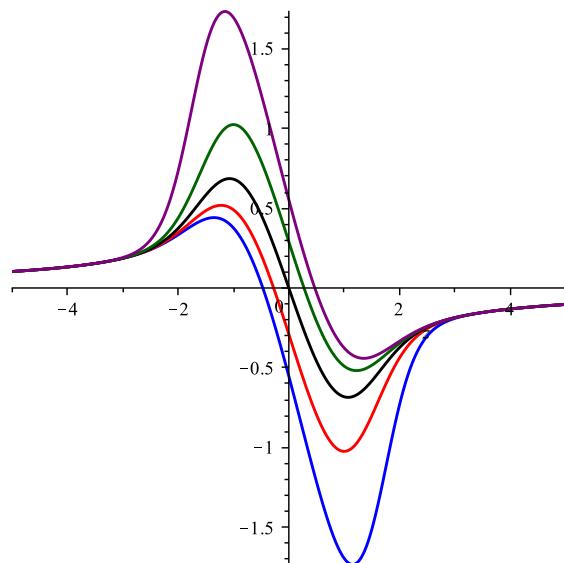


$\sigma_{10,2j}(z), \ j = 1, 2, 3, 4$

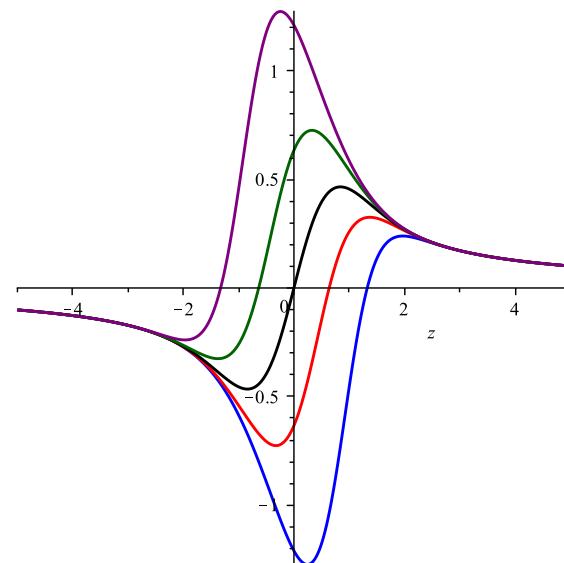
Plots of Bounded Special Function Solutions of S_{IV}

$$\sigma_{\nu,n}(z) = -2nz + \frac{d}{dz} \ln \mathcal{W}(\varphi_\nu, \varphi'_\nu, \dots, \varphi_\nu^{(n-1)})$$

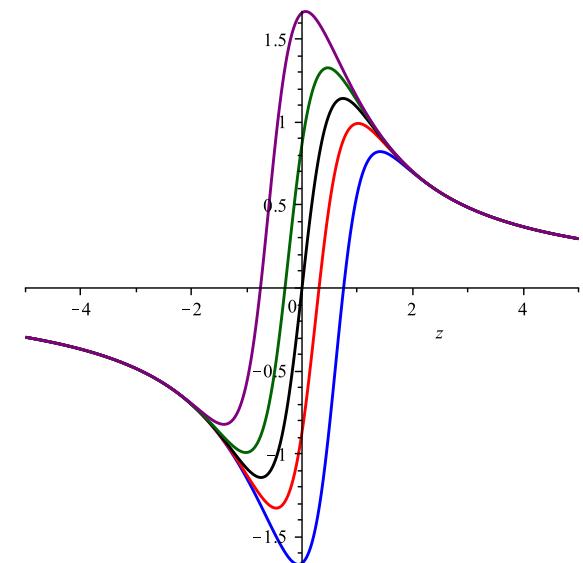
$$\varphi_\nu(z) = \left\{ C_1 D_{-\nu}(\sqrt{2} z) + C_2 D_{-\nu}(-\sqrt{2} z) \right\} \exp\left(\frac{1}{2}z^2\right), \quad C_1 C_2 > 0, \quad \nu > 0$$



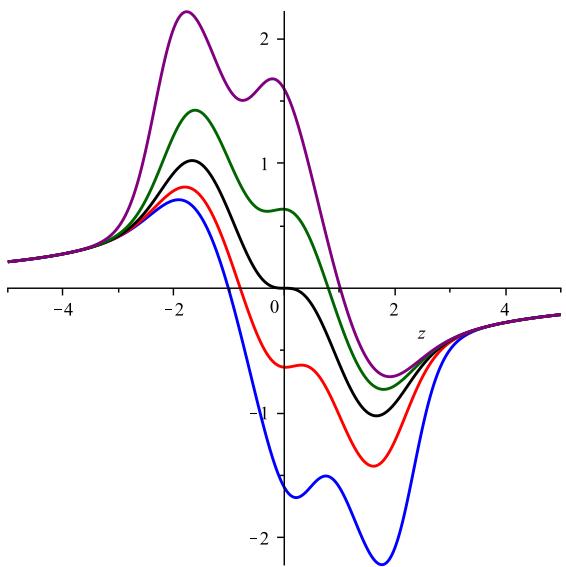
$\sigma_{1/2,1}(z)$



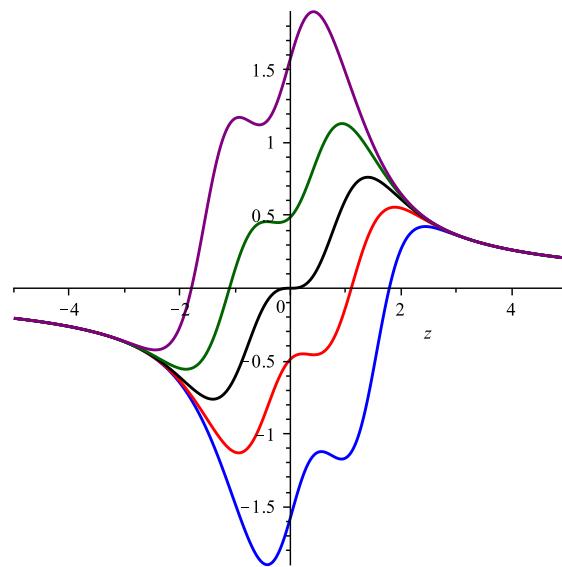
$\sigma_{3/2,1}(z)$



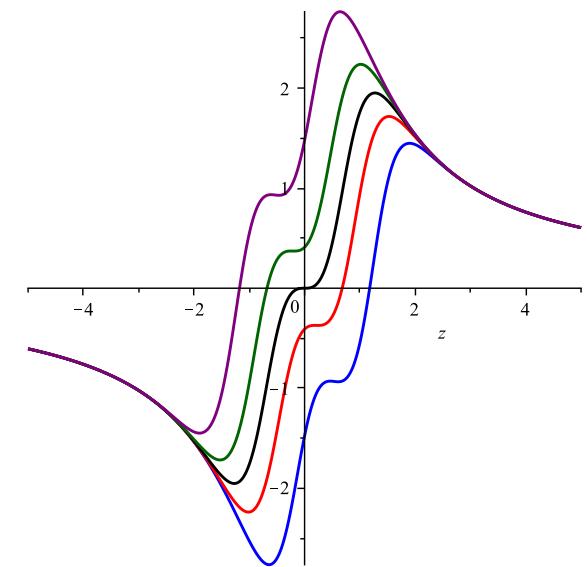
$\sigma_{5/2,1}(z)$



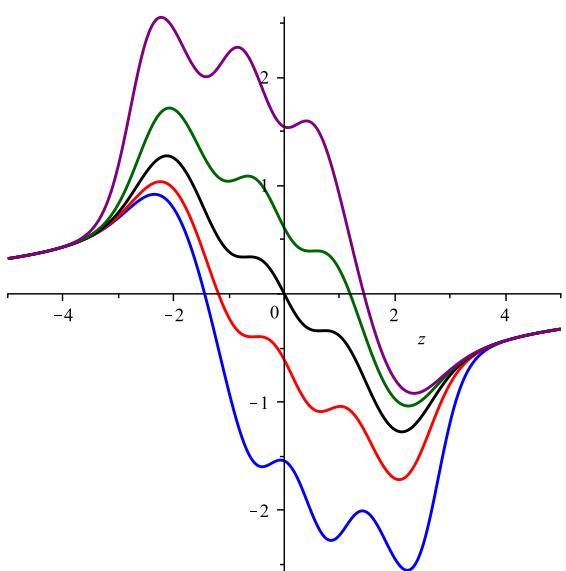
$$\sigma_{1/2,2}(z)$$



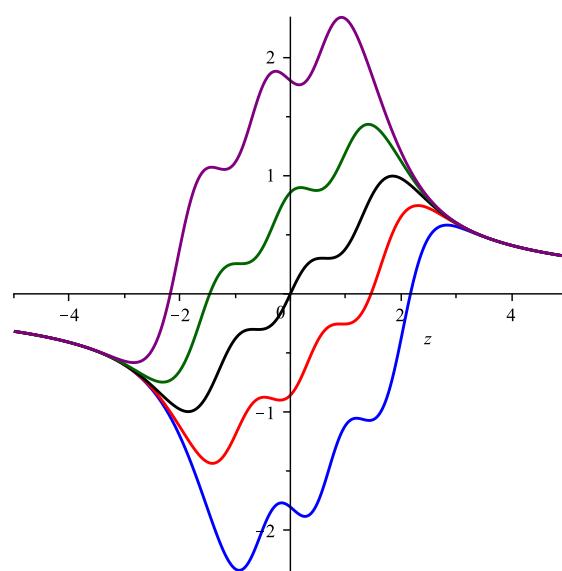
$$\sigma_{3/2,2}(z)$$



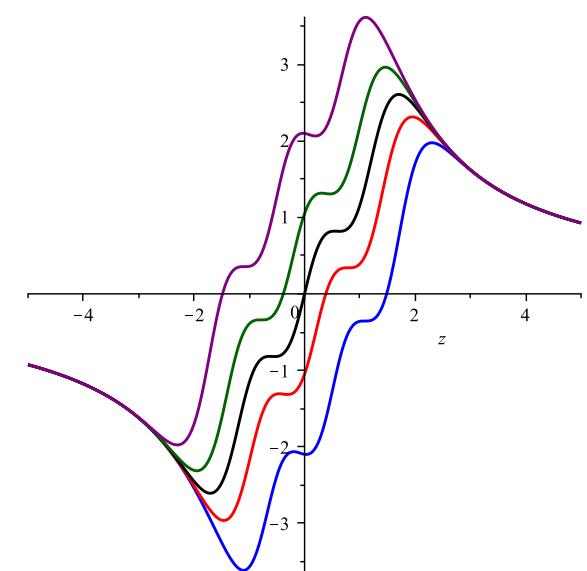
$$\sigma_{5/2,2}(z)$$



$$\sigma_{1/2,3}(z)$$



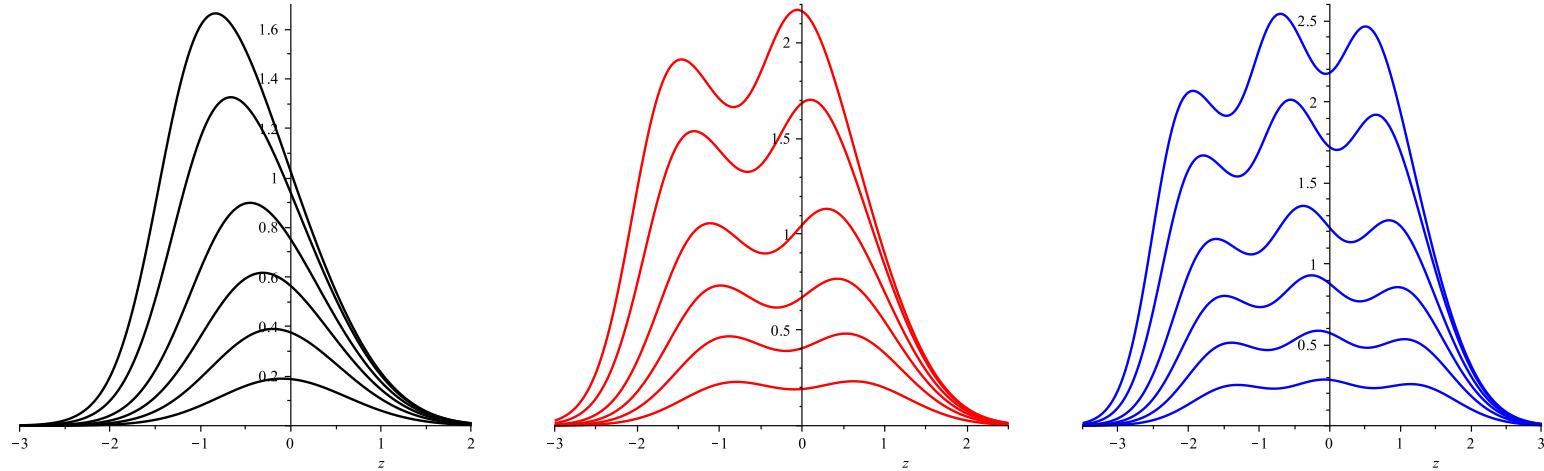
$$\sigma_{3/2,3}(z)$$



$$\sigma_{5/2,3}(z)$$

Plots of Error Function Solutions of S_{IV}

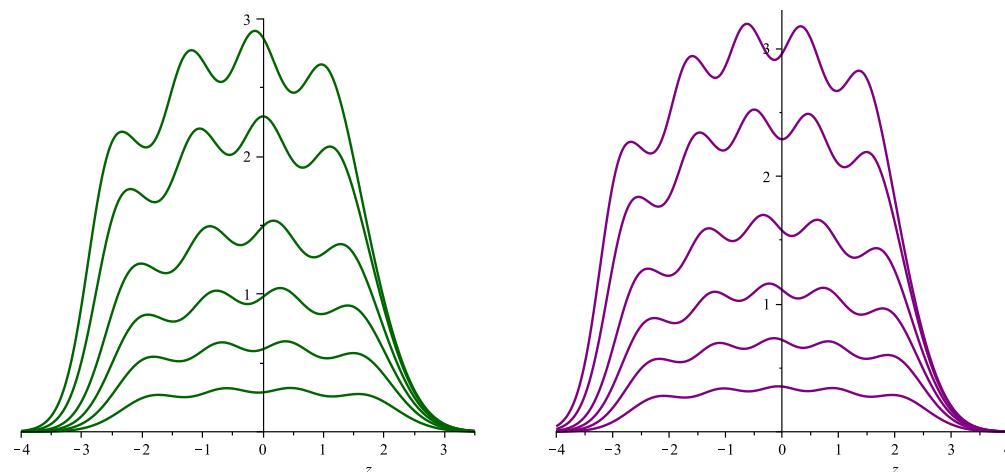
$$\sigma_{m,n} = \frac{d}{dz} \ln \mathcal{W}(\varphi_m, \varphi'_m, \dots, \varphi_m^{(n-1)}), \quad \varphi_m = \exp(-z^2) \frac{d^m}{dz^m} \{C_1 + C_2 \operatorname{erfc}(z)\} \exp(z^2)$$



$\sigma_{1,0}(z)$

$\sigma_{2,1}(z)$

$\sigma_{3,2}(z)$



$\sigma_{4,3}(z)$

$\sigma_{5,4}(z)$

Semi-classical Laguerre Weight

$$\omega(x; t) = x^\nu \exp(-x^2 + tx), \quad x \in \mathbb{R}^+, \quad \nu > -1$$

- **P A Clarkson & K Jordaan**, “The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation”, *Constr. Approx.*, **39** (2014) 223–254

Semi-classical Laguerre weight

Consider monic orthogonal polynomials with respect to the **semi-classical Laguerre weight**

$$\omega(x; t) = x^\nu \exp(-x^2 + tx), \quad x \in \mathbb{R}^+, \quad \nu > -1 \quad (1)$$

which satisfy the three-term recurrence relation

$$xP_n(x; t) = P_{n+1}(x; t) + \alpha_n(t)P_n(x; t) + \beta_n(t)P_{n-1}(x; t) \quad (2)$$

Theorem

(Filipuk, van Assche & Zhang [2012])

The coefficient $\alpha_n(t)$ in the three-term recurrence relation (2) associated with the semi-classical Laguerre weight (1) is given by

$$\alpha_n(t) = \frac{1}{2}q_n\left(\frac{1}{2}t\right) + \frac{1}{2}t$$

where $q_n(z)$ satisfies

$$\frac{d^2q_n}{dz^2} = \frac{1}{2q_n} \left(\frac{dq_n}{dz} \right)^2 + \frac{3}{2}q_n^3 + 4zq_n^2 + 2(z^2 - 2n - 1 - \nu)q_n - \frac{2\nu^2}{q_n} \quad (3)$$

which is P_{IV} with parameters

$$(A, B) = (2n + 1 + \nu, -2\nu^2) \quad (4)$$

- **Filipuk, van Assche & Zhang [2012]** did **not** specify the specific solution of (3).
- The parameters (4) satisfy the condition for P_{IV} to have solutions expressible in terms of **parabolic cylinder functions**.

Theorem

(PAC & Jordaan [2014])

For the semi-classical Laguerre weight

$$\omega(x; t) = x^\nu \exp(-x^2 + tx)$$

the moment $\mu_0(t; \nu)$ is given by

$$\mu_0(t; \nu) = \begin{cases} \frac{\Gamma(\nu + 1) \exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}} D_{-\nu-1} \left(-\frac{1}{2}\sqrt{2}t \right), & \text{if } \nu \neq n \in \mathbb{N} \\ \frac{1}{2}\sqrt{\pi} \frac{d^n}{dt^n} \left\{ \exp \left(\frac{1}{4}t^2 \right) [1 + \operatorname{erf}(\frac{1}{2}t)] \right\}, & \text{if } \nu = n \in \mathbb{N} \end{cases}$$

with $D_\nu(\zeta)$ the parabolic cylinder function and $\operatorname{erf}(z)$ the error function.

Proof. The **parabolic cylinder function** $D_\nu(\zeta)$ has the integral representation

$$D_\nu(\zeta) = \frac{\exp(-\frac{1}{4}\zeta^2)}{\Gamma(-\nu)} \int_0^\infty s^{-\nu-1} \exp(-\frac{1}{2}s^2 - \zeta s) ds$$

Hence for the semi-classical Laguerre weight the moment $\mu_0(t; \nu)$ is given by

$$\begin{aligned} \mu_0(t; \nu) &= \int_0^\infty x^\nu \exp(-x^2 + tx) dx \\ &= 2^{-(\nu+1)/2} \int_0^\infty s^\nu \exp \left(-\frac{1}{2}s^2 + \frac{1}{2}\sqrt{2}ts \right) ds \\ &= \frac{\Gamma(\nu + 1) \exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}} D_{-\nu-1} \left(-\frac{1}{2}\sqrt{2}t \right) \end{aligned}$$

If $\nu = n \in \mathbb{N}$, then

$$D_{-n-1}(\zeta) = \sqrt{\frac{\pi}{2}} \frac{(-1)^n}{n!} \exp\left(-\frac{1}{4}\zeta^2\right) \frac{d^n}{d\zeta^n} \left\{ \exp\left(\frac{1}{2}\zeta^2\right) \operatorname{erfc}\left(\frac{1}{2}\sqrt{2}\zeta\right) \right\},$$

with $\operatorname{erfc}(z)$ the complementary error function. Since $\operatorname{erfc}(-z) = 1 - \operatorname{erf}(z)$, then

$$\mu_0(t; n) = \frac{1}{2}\sqrt{\pi} \frac{d^n}{dt^n} \left\{ \exp\left(\frac{1}{4}t^2\right) [1 + \operatorname{erf}\left(\frac{1}{2}t\right)] \right\}$$

Corollary

The moment $\mu_0(t; \nu)$ satisfies the ordinary differential equation

$$\frac{d^2\mu_0}{dt^2} - \frac{1}{2}t \frac{d\mu_0}{dt} - \frac{1}{2}(\nu + 1)\mu_0 = 0 \tag{1}$$

Proof. The parabolic cylinder function $D_\nu(\zeta)$ satisfies

$$\frac{d^2 D_\nu}{d\zeta^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}\zeta^2\right) D_\nu = 0$$

and so it follows from its definition that the moment $\mu_0(t; \nu)$ satisfies equation (??).

Theorem

(PAC & Jordaan [2014])

The coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the three-term recurrence relation

$$xP_n(x; t) = P_{n+1}(x; t) + \alpha_n(t)P_n(x; t) + \beta_n(t)P_{n-1}(x; t)$$

associated with the semi-classical Laguerre weight

$$\omega(x; t) = x^\nu \exp(-x^2 + tx), \quad x \in \mathbb{R}^+, \quad \nu > -1$$

are given by

$$\alpha_n(t) = \frac{d}{dt} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \quad \beta_n(t) = \frac{\Delta_{n+1}(t)\Delta_{n-1}(t)}{\Delta_n^2(t)}, \quad n \geq 0$$

where $\Delta_n(t)$ is the Hankel determinant given by

$$\Delta_n(t) = \mathcal{W} \left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right), \quad n \geq 1$$

$\Delta_0(t) = 1$ and $\Delta_{-1}(t) = 0$, with

$$\mu_0(t; \nu) = \begin{cases} \frac{\Gamma(\nu + 1) \exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}} D_{-\nu-1} \left(-\frac{1}{2}\sqrt{2}t \right), & \text{if } \nu \neq n \in \mathbb{N} \\ \frac{1}{2}\sqrt{\pi} \frac{d^n}{dt^n} \left\{ \exp \left(\frac{1}{4}t^2 \right) [1 + \operatorname{erf}(\frac{1}{2}t)] \right\}, & \text{if } \nu = n \in \mathbb{N} \end{cases}$$

$D_\nu(\zeta)$ the parabolic cylinder function and $\operatorname{erf}(z)$ the error function.

Remarks:

- The Hankel determinant $\Delta_n(t)$ satisfies the **Toda equation**

$$\frac{d^2}{dt^2} \ln \Delta_n(t) = \frac{\Delta_{n-1}(t)\Delta_{n+1}(t)}{\Delta_n^2(t)}$$

and the fourth-order, bi-linear equation

$$\begin{aligned} \Delta_n \frac{d^4 \Delta_n}{dt^4} - 4 \frac{d^3 \Delta_n}{dt^3} \frac{d \Delta_n}{dt} + 3 \left(\frac{d^2 \Delta_n}{dt^2} \right)^2 - (\frac{1}{4}t^2 + 4n + 2\nu) \left\{ \Delta_n \frac{d^2 \Delta_n}{dt^2} - \left(\frac{d \Delta_n}{dt} \right)^2 \right\} \\ + \frac{1}{4}t \Delta_n \frac{d \Delta_n}{dt} + \frac{1}{2}n(n+\nu) \Delta_n^2 = 0 \end{aligned}$$

- The function $S_n(t) = \frac{d}{dt} \ln \Delta_n(t)$ satisfies

$$4 \left(\frac{d^2 S_n}{dt^2} \right)^2 - \left(t \frac{d S_n}{dt} - S_n \right)^2 + 4 \frac{d S_n}{dt} \left(2 \frac{d S_n}{dt} - n \right) \left(2 \frac{d S_n}{dt} - n - \nu \right) = 0$$

which is equivalent to S_{IV} , the P_{IV} σ -equation (let $S_n(t) = \frac{1}{2}\sigma(z)$, with $z = 2t$), so

$$\alpha_n(t) = \frac{d}{dt} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)} = S_{n+1}(t) - S_n(t), \quad \beta_n(t) = \frac{d^2}{dt^2} \ln \Delta_n(t) = \frac{d S_n}{dt}$$

The first few coefficients in the recurrence relation are given by

$$\begin{aligned}\alpha_0(t) &= \frac{1}{2}t - \frac{D_{-\nu}(-\frac{1}{2}\sqrt{2}t)}{D_{-\nu-1}(-\frac{1}{2}\sqrt{2}t)} \equiv \Psi_\nu(t) \\ \alpha_1(t) &= \frac{1}{2}t - \Psi_\nu(t) - \frac{\Psi_\nu(t)}{2\Psi_\nu^2(t) - t\Psi_\nu(t) - \nu - 1} \\ \alpha_2(t) &= \frac{1}{2}t + \frac{2\nu + 4}{t} + \frac{\Psi_\nu(t)}{2\Psi_\nu^2(t) - t\Psi_\nu(t) - \nu - 1} \\ &\quad - \frac{2[(\nu + 1)t^2 + 4(\nu + 2)(2\nu + 3)]\Psi_\nu^2(t) - (\nu + 1)t[t^2 + 2(4\nu + 9)]\Psi_\nu(t)}{2t[2t\Psi_\nu^3(t) - (t^2 - 4\nu - 6)\Psi_\nu^2(t) - 3(\nu + 1)t\Psi_\nu(t) - 2(\nu + 1)^2]} \\ &\quad + \frac{(\nu + 1)^2[t^2 + 8(\nu + 2)]}{2t[2t\Psi_\nu^3(t) - (t^2 - 4\nu - 6)\Psi_\nu^2(t) - 3(\nu + 1)t\Psi_\nu(t) - 2(\nu + 1)^2]}\end{aligned}$$

$$\beta_1(t) = -\Psi_\nu^2(t) + \frac{1}{2}t\Psi_\nu(t) + \frac{1}{2}(\nu + 1),$$

$$\beta_2(t) = -\frac{2t\Psi_\nu^3(t) - (t^2 - 4\nu - 6)\Psi_\nu^2(t) - 3(\nu + 1)t\Psi_\nu(t) - 2(\nu + 1)^2}{2[\Psi_\nu^2(t) - \frac{1}{2}t\Psi_\nu(t) - \frac{1}{2}(\nu + 1)]^2}$$

Hence, using the three-term recurrence relation

$$P_{n+1}(x; t) = [x - \alpha_n(t)]P_n(x; t) - \beta_n(t)P_{n-1}(x; t), \quad n = 0, 1, 2, \dots$$

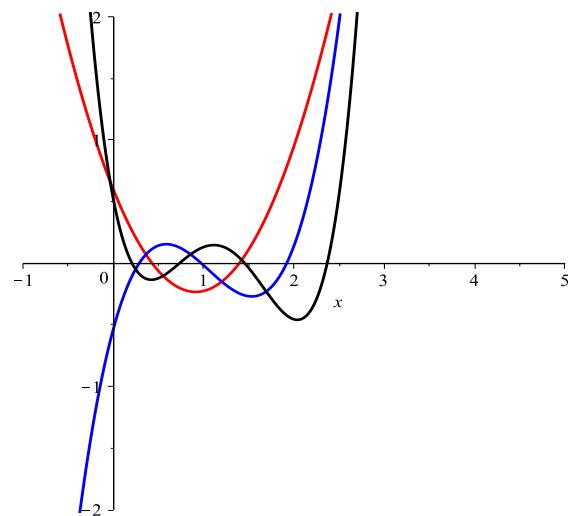
with $P_0(x; t) = 1$ and $P_{-1}(x; t) = 0$, then the first few polynomials are given by

$$P_1(x; t) = x - \Psi_\nu$$

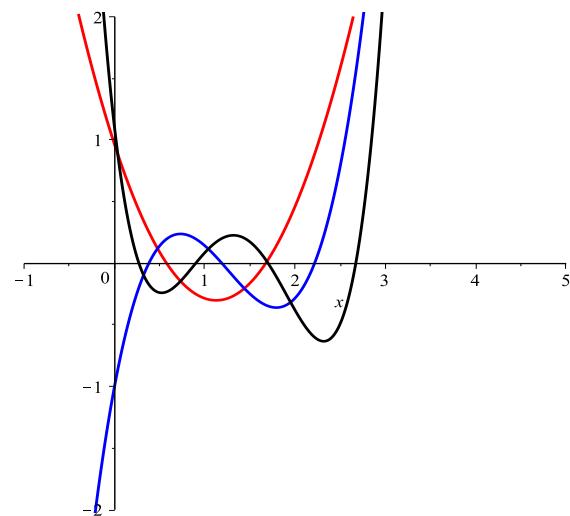
$$P_2(x; t) = x^2 - \frac{2t\Psi_\nu^2 - (t^2 + 2)\Psi_\nu - (\nu + 1)t}{2[\Psi_\nu^2 - \frac{1}{2}t\Psi_\nu - \frac{1}{2}(\nu + 1)]}x - \frac{2(\nu + 2)\Psi_\nu^2 - (\lambda + 1)\Psi_\nu - (\lambda + 1)^2}{2[\Psi_\nu^2 - \frac{1}{2}t\Psi_\nu - \frac{1}{2}(\nu + 1)]}$$

$$\begin{aligned} P_3(x; t) &= x^3 - \left\{ \frac{4(t^2 + 2\nu + 4)\Psi_\nu^3 - 2t(t^2 - \nu - 1)\Psi_\nu^2}{2[2t\Psi_\nu^3 - (t^2 - 4\nu - 6)\Psi_\nu^2 - 3(\nu + 1)t\Psi_\nu - 2(\nu + 1)^2]} \right. \\ &\quad \left. - \frac{(\nu + 1)(5t^2 + 4\lambda + 6)\Psi_\nu + 3(\nu + 1)^2t}{2[2t\Psi_\nu^3 - (t^2 - 4\nu - 6)\Psi_\nu^2 - 3(\nu + 1)t\Psi_\nu - 2(\nu + 1)^2]} \right\} x^2 \\ &\quad + \left\{ \frac{2t(t^2 + 2\nu + 4)\Psi_\nu^3 - [t^4 + 4(2\nu + 5)(\nu + 2)]\Psi_\nu^2}{4[2t\Psi_\nu^3 - (t^2 - 4\nu - 6)\Psi_\nu^2 - 3(\nu + 1)t\Psi_\nu - 2(\nu + 1)^2]} \right. \\ &\quad \left. - \frac{2(\nu + 1)t(t^2 - \nu - 5)\Psi_\nu + (\nu + 1)^2(t^2 - 4\nu - 12)}{4[2t\Psi_\nu^3 - (t^2 - 4\nu - 6)\Psi_\nu^2 - 3(\nu + 1)t\Psi_\nu - 2(\nu + 1)^2]} \right\} x \\ &\quad + \frac{2[(\nu + 1)t^2 + 4(\nu + 2)^2]\Psi_\nu^3 - (\nu + 1)t(t^2 + 2\nu + 8)\Psi_\nu^2}{4[2t\Psi_\nu^3 - (t^2 - 4\nu - 6)\Psi_\nu^2 - 3(\nu + 1)t\Psi_\nu - 2(\nu + 1)^2]} \\ &\quad - \frac{2(\nu + 1)^2(t^2 + 2\nu + 5)\Psi_\nu + (\nu + 1)^3t}{4[2t\Psi_\nu^3 - (t^2 - 4\nu - 6)\Psi_\nu^2 - 3(\nu + 1)t\Psi_\nu - 2(\nu + 1)^2]} \end{aligned}$$

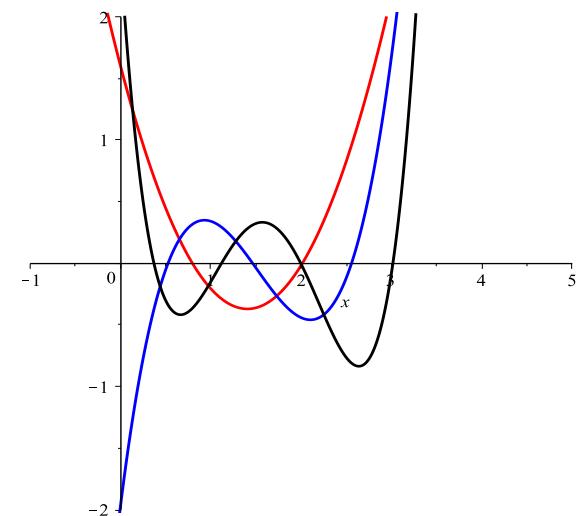
$$P_2(x; t) \quad P_3(x; t) \quad P_4(x; t)$$



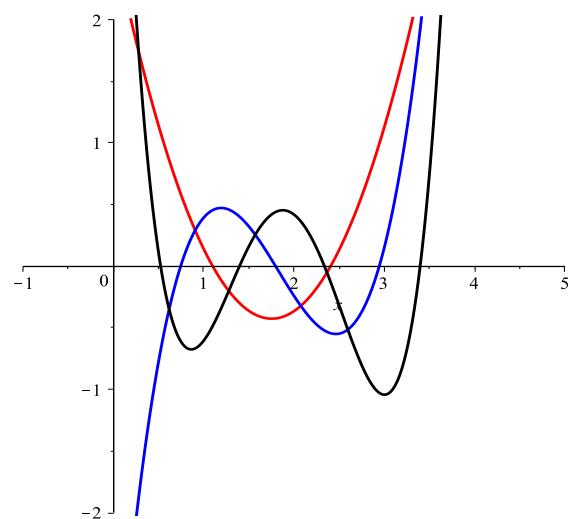
$$t = 0$$



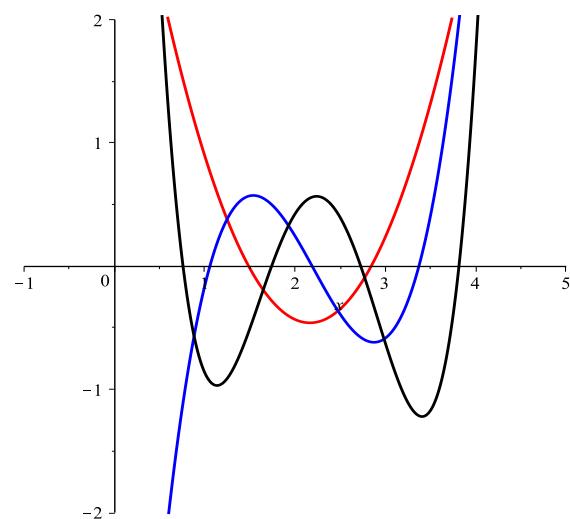
$$t = 1$$



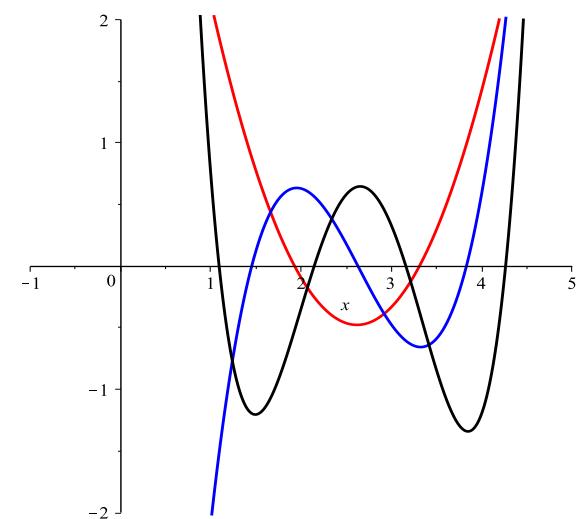
$$t = 2$$



$$t = 3$$



$$t = 4$$



$$t = 5$$

Semi-classical Hermite Weight

$$\omega(x; t) = |x|^\nu \exp(-x^2 + tx), \quad x \in \mathbb{R}, \quad \nu > -1$$

- **P A Clarkson & K Jordaan**, “The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation”, *Constr. Approx.*, **39** (2014) 223–254

Semi-classical Hermite weight

Consider the **semi-classical Hermite weight**

$$\omega(x; t) = |x|^\nu \exp(-x^2 + tx), \quad x \in \mathbb{R}, \quad \nu > -1$$

- The moment $\mu_k(t; \nu)$ is given by

$$\begin{aligned} \mu_k(t; \nu) &= \int_{-\infty}^{\infty} x^k |x|^\nu \exp(-x^2 + tx) dx \\ &= \frac{d^k}{dt^k} \left(\int_{-\infty}^{\infty} |x|^\nu \exp(-x^2 + tx) dx \right) = \frac{d^k \mu_0}{dt^k} \end{aligned}$$

- The Hankel determinant $\Delta_n(t)$ is given by

$$\Delta_n(t) = \det \left[\mu_{j+k}(t) \right]_{j,k=0}^{n-1} \equiv \mathcal{W} \left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right)$$

where

$$\mu_0(t; \nu) = \begin{cases} \frac{\Gamma(\nu + 1) \exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}} \left\{ D_{-\nu-1} \left(-\frac{1}{2}\sqrt{2}t \right) + D_{-\nu-1} \left(\frac{1}{2}\sqrt{2}t \right) \right\}, & \nu \notin \mathbb{N} \\ \sqrt{\pi} \left(-\frac{1}{2}i \right)^{2N} H_{2N} \left(\frac{1}{2}it \right) \exp \left(\frac{1}{4}t^2 \right), & \nu = 2N \\ \sqrt{\pi} \frac{d^{2N+1}}{dt^{2N+1}} \left\{ \operatorname{erf} \left(\frac{1}{2}t \right) \exp \left(\frac{1}{4}t^2 \right) \right\}, & \nu = 2N + 1 \end{cases}$$

Theorem

(PAC & Jordaan [2014])

The recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the three-term recurrence relation

$$xP_n(x; t) = P_{n+1}(x; t) + \alpha_n(t)P_n(x; t) + \beta_n(t)P_{n-1}(x; t),$$

for monic polynomials orthogonal with respect to the semi-classical Hermite weight

$$\omega(x; t) = |x|^\nu \exp(-x^2 + tx), \quad x \in \mathbb{R}, \quad \nu > -1$$

are given by

$$\alpha_n(t) = \frac{d}{dt} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \quad \beta_n(t) = \frac{d^2}{dt^2} \ln \Delta_n(t)$$

where $\Delta_n(t)$ is the Hankel determinant

$$\Delta_n(t) = \det \left[\mu_{j+k}(t) \right]_{j,k=0}^{n-1} \equiv \mathcal{W} \left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right)$$

with

$$\mu_0(t; \nu) = \begin{cases} \frac{\Gamma(\nu + 1) \exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}} \left\{ D_{-\nu-1} \left(-\frac{1}{2}\sqrt{2}t \right) + D_{-\nu-1} \left(\frac{1}{2}\sqrt{2}t \right) \right\}, & \nu \notin \mathbb{N} \\ \sqrt{\pi} \left(-\frac{1}{2}i \right)^{2N} H_{2N} \left(\frac{1}{2}it \right) \exp \left(\frac{1}{4}t^2 \right), & \nu = 2N \\ \sqrt{\pi} \frac{d^{2N+1}}{dt^{2N+1}} \left\{ \operatorname{erf} \left(\frac{1}{2}t \right) \exp \left(\frac{1}{4}t^2 \right) \right\}, & \nu = 2N + 1 \end{cases}$$

Recurrence coefficients for $\omega(x; t) = x^2 \exp(-x^2 + tx)$

$$\alpha_0(t) = \frac{1}{2}t + \frac{2t}{t^2 + 2}$$

$$\alpha_1(t) = \frac{1}{2}t + \frac{4t^3}{t^4 + 12} - \frac{2t}{t^2 + 2}$$

$$\alpha_2(t) = \frac{1}{2}t + \frac{6t(t^4 + 12 - 4t^2)}{t^6 - 6t^4 + 36t^2 + 72} - \frac{4t^3}{t^4 + 12}$$

$$\alpha_3(t) = \frac{1}{2}t + \frac{8t^3(t^4 + 60 - 12t^2)}{t^8 - 16t^6 + 120t^4 + 720} - \frac{6t(t^4 + 12 - 4t^2)}{t^6 - 6t^4 + 36t^2 + 72}$$

$$\alpha_4(t) = \frac{1}{2}t + \frac{10t(t^8 + 216t^4 + 720 - 24t^6 - 480t^2)}{t^{10} - 30t^8 + 360t^6 - 1200t^4 + 3600t^2 + 7200} - \frac{8t^3(t^4 + 60 - 12t^2)}{t^8 - 16t^6 + 120t^4 + 720}$$

$$\beta_1(t) = \frac{1}{2} - \frac{2(t^2 - 2)}{(t^2 + 2)^2}$$

$$\beta_2(t) = 1 - \frac{4t^2(t^2 - 6)(t^2 + 6)}{(t^4 + 12)^2}$$

$$\beta_3(t) = \frac{3}{2} - \frac{6(t^4 - 12t^2 + 12)(t^6 + 6t^4 + 36t^2 - 72)}{(t^6 - 6t^4 + 36t^2 + 72)^2}$$

$$\beta_4(t) = 2 - \frac{8t^2(t^4 - 20t^2 + 60)(t^8 + 72t^4 - 2160)}{(t^8 - 16t^6 + 120t^4 + 720)^2}$$

Hence, using the three-term recurrence relation

$$P_{n+1}(x; t) = [x - \alpha_n(t)]P_n(x; t) - \beta_n(t)P_{n-1}(x; t), \quad n = 0, 1, 2, \dots$$

with $P_0(x; t) = 1$ and $P_{-1}(x; t) = 0$, then the first few polynomials are given by

$$P_1(x; t) = x - \frac{t(t^2 + 6)}{2(t^2 + 2)}$$

$$P_2(x; t) = x^2 - \frac{t(t^4 + 4t^2 + 12)}{t^4 + 12}x + \frac{t^6 + 6t^4 + 36t^2 - 72}{4(t^4 + 12)}$$

$$\begin{aligned} P_3(x; t) &= x^3 - \frac{3t(t^6 - 2t^4 + 20t^2 + 120)}{2(t^6 - 6t^4 + 36t^2 + 72)}x^2 + \frac{3(t^8 + 40t^4 - 240)}{4(t^6 - 6t^4 + 36t^2 + 72)}x \\ &\quad - \frac{t(t^8 + 72t^4 - 2160)}{8(t^6 - 6t^4 + 36t^2 + 72)} \end{aligned}$$

$$\begin{aligned} P_4(x; t) &= x^4 - \frac{2t(t^8 - 12t^6 + 72t^4 + 240t^2 + 720)}{t^8 - 16t^6 + 120t^4 + 720}x^3 \\ &\quad + \frac{3(t^{10} - 10t^8 + 80t^6 + 1200t^2 - 2400)}{2(t^8 - 16t^6 + 120t^4 + 720)}x^2 \\ &\quad - \frac{t(t^{10} - 10t^8 + 120t^6 - 240t^4 - 1200t^2 - 7200)}{2(t^8 - 16t^6 + 120t^4 + 720)}x \\ &\quad + \frac{t^{12} - 12t^{10} + 180t^8 - 480t^6 - 3600t^4 - 43200t^2 + 43200}{16(t^8 - 16t^6 + 120t^4 + 720)} \end{aligned}$$

Special function solutions of Painlevé equations

Number of (essential) parameters	Special function	Number of parameters	Associated orthogonal polynomial	Number of parameters
P _I 0	—			
P _{II} 1	Airy Ai(z), Bi(z)	0	—	
P _{III} 2	Bessel J _ν (z), Y _ν (z), I _ν (z), K _ν (z)	1	—	
P _{IV} 2	Parabolic cylinder D _ν (z)	1	Hermite H _n (z)	0
P _V 3	Kummer M(a, b, z), U(a, b, z) Whittaker M _{κ,μ} (z), W _{κ,μ} (z)	2	Associated Laguerre L _n ^(k) (z)	1
P _{VI} 4	hypergeometric 2F ₁ (a, b; c; z)	3	Jacobi P _n ^(α,β) (z)	2

Examples Associated with the Third Painlevé Equation

$$\frac{d^2q}{dt^2} = \frac{1}{q} \left(\frac{dq}{dt} \right)^2 - \frac{1}{\zeta} \frac{dq}{dt} + \frac{Aq^2}{4t^2} + \frac{B}{4t} + \frac{Cq^3}{4t^2} + \frac{D}{4q} \quad P_{III'}$$

$$\left(t \frac{d^2\sigma}{dt^2} \right)^2 + \left\{ 4 \left(\frac{d\sigma}{dt} \right)^2 - 1 \right\} \left(t \frac{d\sigma}{dt} - \sigma \right) + \vartheta_0 \vartheta_\infty \frac{d\sigma}{dt} = \frac{1}{4} (\vartheta_0^2 + \vartheta_\infty^2) \quad S_{III'}$$

Perturbed Laguerre weight

Consider orthogonal polynomials with respect to the **perturbed Laguerre weight**

$$\omega(x; z) = x^\nu \exp(-x - t/x), \quad x \in \mathbb{R}^+, \quad \nu > 0$$

Define the Hankel determinant

$$\Delta_n(t) = \det \left[\mu_{j+k}(t) \right]_{j,k=0}^{n-1}, \quad \mu_k(t) = \int_0^\infty x^{\nu+k} \exp(-x - t/x) dx$$

Then **Chen & Its [2010]** show that

$$H_n(t) = t \frac{d}{dt} \ln \Delta_n(t)$$

satisfies

$$\left(t \frac{d^2 H_n}{dt^2} \right)^2 = \left[(2n + \nu) \frac{d H_n}{dt} - n \right]^2 - 4 \frac{d H_n}{dt} \left(\frac{d H_n}{dt} - 1 \right) \left[t \frac{d H_n}{dt} - H_n + n(n + \nu) \right]$$

which is equivalent to a special case of $S_{III'}$, the $P_{III'}$ σ -equation. Specifically, letting

$$H_n(t) = \sigma + \frac{1}{2}t + \frac{1}{4}n^2 - \frac{1}{2}n(\nu + 1) - \frac{1}{4}\nu^2$$

yields

$$\left(t \frac{d^2 \sigma}{dt^2} \right)^2 + \left\{ 4 \left(\frac{d \sigma}{dt} \right)^2 - 1 \right\} \left(t \frac{d \sigma}{dt} - \sigma \right) + (n^2 - \nu^2) \frac{d \sigma}{dt} = \frac{1}{2}(n^2 + \nu^2)$$

which is $S_{III'}$ with $(\vartheta_0, \vartheta_\infty) = (n + \nu, n - \nu)$.

For the **perturbed Laguerre weight**

$$\omega(x; t) = x^\nu \exp(-x - t/x), \quad x \in \mathbb{R}^+, \quad \nu > 0, \quad t > 0$$

the associated moments are

$$\mu_k(t) = \int_0^\infty x^{\nu+k} \exp(-x - t/x) dx = 2t^{(\nu+k+1)/2} K_{\nu+k+1}(2\sqrt{t})$$

with $K_\nu(z)$ the **modified Bessel function**. Hence the Hankel determinant is given by

$$\begin{aligned} \Delta_n(t) &= \det \left[\mu_{j+k}(t) \right]_{j,k=0}^{n-1} = 2^n t^{n(n+\nu)/2} \det \left[K_{\nu+j+k+1}(2\sqrt{t}) \right]_{j,k=0}^{n-1} \\ &= 2^n t^{n(\nu+1)/2} \det \left[\left(t \frac{d}{dt} \right)^{j+k} K_{\nu+n}(2\sqrt{t}) \right]_{j,k=0}^{n-1} \end{aligned}$$

using properties of Bessel functions. Then

$$H_n(t) = t \frac{d}{dt} \ln \Delta_n(t)$$

satisfies

$$\left(t \frac{d^2 H_n}{dt^2} \right)^2 = \left[n - (2n + \nu) \frac{d H_n}{dt} \right]^2 - 4 \frac{d H_n}{dt} \left(\frac{d H_n}{dt} - 1 \right) \left[t \frac{d H_n}{dt} - H_n + n(n + \nu) \right]$$

which is equivalent to a special case of $S_{III'}$, the $P_{III'}$ σ -equation.

Theorem

(Okamoto [1987]; Forrester & Witte [2002])

Special function solutions of the P_{III'} σ-equation

$$\left(t \frac{d^2\sigma}{dt^2}\right)^2 + \left\{4 \left(\frac{d\sigma}{dt}\right)^2 - 1\right\} \left(t \frac{d\sigma}{dt} - \sigma\right) + \vartheta_0 \vartheta_\infty \frac{d\sigma}{dt} = \frac{1}{4}(\vartheta_0^2 + \vartheta_\infty^2) \quad S_{III'}$$

are given by

$$\sigma(t) = t \frac{d}{dt} \ln \tau_{n,\nu}(t) + \frac{1}{2}\varepsilon_1\varepsilon_2 t + \frac{1}{4}\nu^2 + \frac{1}{2}n(1 - \varepsilon_1\nu) - \frac{1}{4}n^2$$

for the parameters $(\vartheta_0, \vartheta_\infty) = (\nu + n, \varepsilon_1\varepsilon_2(\nu - n))$, where $\tau_{n,\nu}(t)$ is the determinant

$$\tau_{n,\nu}(t) = \det \left[\left(t \frac{d}{dt}\right)^{j+k} \psi_\nu(t) \right]_{j,k=0}^{n-1}$$

with $\psi_\nu(t)$ given by

$$\psi_\nu(t) = \begin{cases} t^{\nu/2} \{C_1 J_\nu(2\sqrt{t}) + C_2 Y_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = 1 \\ t^{-\nu/2} \{C_1 J_\nu(2\sqrt{t}) + C_2 Y_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = -1 \\ t^{\nu/2} \{C_1 I_\nu(2\sqrt{t}) + C_2 K_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = -1 \\ t^{-\nu/2} \{C_1 I_\nu(2\sqrt{t}) + C_2 K_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = 1 \end{cases}$$

C_1 and C_2 arbitrary constants, $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$ Bessel functions.

Examples Associated with the Fifth Painlevé Equation

$$\frac{d^2q}{dz^2} = \left(\frac{1}{2q} + \frac{1}{q-1} \right) \left(\frac{dq}{dz} \right)^2 - \frac{1}{z} \frac{dq}{dz} + \frac{(q-1)^2}{z^2} \left(Aq + \frac{B}{q} \right) + \frac{Cq}{z} + \frac{Dq(q+1)}{q-1}$$

P_V

$$\left(z \frac{d^2\sigma}{dz^2} \right)^2 = \left[2 \left(\frac{d\sigma}{dz} \right)^2 - z \frac{d\sigma}{dz} + \sigma \right]^2 - 4 \prod_{j=1}^4 \left(\frac{d\sigma}{dz} + \kappa_j \right)$$

S_V

The **Kummer functions** $M(a, b, z)$ and $U(a, b, z)$ have the integral representations

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zs} s^{a-1} (1-s)^{b-a-1} ds$$

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zs} s^{a-1} (1+s)^{b-a-1} ds$$

- For the **perturbed Jacobi weight** (**Basor, Chen & Ehrhardt [2010]**)

$$\omega(x; z) = x^{\alpha-1} (1+x)^{\beta-1} e^{-zx}, \quad x \in [0, 1], \quad \alpha > 0, \quad \beta > 0$$

the moments are given by

$$\mu_0(z; \alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} e^{-z} M(\alpha, \alpha+\beta, z),$$

$$\mu_k(z; \alpha, \beta) = (-1)^k \frac{d^k}{dz^k} \mu_0(z; \alpha, \beta)$$

- For the **Pollaczek-Jacobi weight** (**Chen & Dai [2010]**)

$$\omega(x; z) = x^{\alpha-1} (1-x)^{\beta-1} e^{-z/x}, \quad x \in [0, 1], \quad \alpha > 0, \quad \beta > 0$$

the k th moment is

$$\mu_k(z; \alpha, \beta) = \Gamma(\beta) e^{-z} U(\beta, 1-\alpha-k, z)$$

For both these weights $H_n(z) = z \frac{d}{dz} \ln \Delta_n(z)$, with $\Delta_n(z) = \det \left[\mu_{j+k}(z) \right]_{j,k=0}^{n-1}$, satisfies an equation which is equivalent to a special case of S_V , the P_V σ -equation.

Theorem

(Okamoto [1987]; Forrester & Witte [2002])

Special function solutions of the P_V σ-equation

$$\left(z \frac{d^2\sigma}{dz^2}\right)^2 - \left\{2 \left(\frac{d\sigma}{dz}\right)^2 - z \frac{d\sigma}{dz} + \sigma\right\}^2 + 4 \prod_{j=1}^4 \left(\frac{d\sigma}{dz} + \kappa_j\right) = 0 \quad S_V$$

are given by

$$\begin{aligned} \sigma(z) = z \frac{d}{dz} \ln \mathcal{W}_n(\varphi_{\alpha,\beta}) - \frac{1}{4}(3n + 2\alpha - \beta - 1)z \\ - \frac{5}{8}n^2 - \frac{1}{4}(2\alpha - 3\beta - 1)n - \frac{1}{8}(2\alpha - \beta - 1)^2 \end{aligned}$$

for the parameters

$$\begin{aligned} \kappa_1 &= \frac{1}{4}(2\alpha - \beta - n - 1), & \kappa_3 &= \frac{1}{4}(2\alpha - \beta + 3n - 1) \\ \kappa_2 &= -\frac{1}{4}(2\alpha + \beta + n - 3), & \kappa_4 &= -\frac{1}{4}(2\alpha - 3\beta + n + 1) \end{aligned}$$

where $\mathcal{W}_n(\varphi_{\alpha,\beta})$ is the determinant given by

$$\mathcal{W}_n(\varphi_{\alpha,\beta}) = \det \left[\left(z \frac{d}{dz} \right)^{j+k} \varphi_{\alpha,\beta}(z) \right]_{j,k=0}^{n-1}$$

with

$$\varphi_{\alpha,\beta}(z) = C_1 M(\alpha, \beta, z) + C_2 U(\alpha, \beta, z)$$

C_1 and C_2 arbitrary constants, $M(\alpha, \beta, z)$ and $U(\alpha, \beta, z)$ **Kummer functions**.

Deformed Laguerre weight

Consider orthogonal polynomials with the respect to the **deformed Laguerre weight**

$$\omega(x; z) = x^\nu (x + z)^\lambda e^{-x}, \quad x \in \mathbb{R}^+, \quad \nu > 0, \quad \lambda > 0$$

Define the Hankel determinant

$$\Delta_n(z; \nu, \lambda) = \det \left[\mu_{j+k}(z; \nu, \lambda) \right]_{j,k=0}^{n-1}$$

where

$$\mu_k(z; \nu, \lambda) = \int_0^\infty x^{\nu+k} (x + z)^\lambda e^{-x} dx$$

Chen & McKay [2012] (also **Basor, Chen & McKay [2013]**) show that

$$H_n(z; \nu, \lambda) = z \frac{d}{dz} \ln \Delta_n(z; \nu, \lambda)$$

satisfies

$$\begin{aligned} \left(z \frac{d^2 H_n}{dz^2} \right)^2 &= \left[(z + 2n + \nu + \lambda) \frac{d H_n}{dz} - H_n + n\lambda \right]^2 \\ &\quad - 4 \frac{d H_n}{dz} \left(\frac{d H_n}{dz} + \lambda \right) \left[z \frac{d H_n}{dz} - H_n + n(n + \nu + \lambda) \right] \end{aligned}$$

which is equivalent to a special case of S_V , the P_V σ -equation.

Remarks

- For the **deformed Laguerre weight**

$$\omega(x; z) = x^\nu (x + z)^\lambda e^{-x}, \quad x \in \mathbb{R}^+, \quad \nu > 0, \quad \lambda > 0$$

the k th moment is

$$\begin{aligned} \mu_k(z; \nu, \lambda) &= \int_0^\infty x^{\nu+k} (x + z)^\lambda e^{-x} dx \\ &= z^{\nu+\lambda+k+1} \int_0^\infty s^{\nu+k} (1 + s)^\lambda e^{-sz} dx \\ &= \Gamma(\nu + k + 1) z^{\nu+\lambda+k+1} U(\nu + k + 1, \nu + \lambda + k + 2, t) \end{aligned}$$

with $U(a, b, z)$ the **Kummer function** of the second kind.

- In the special case of the deformed Laguerre weight when $\lambda = m \in \mathbb{Z}^+$ then

$$\begin{aligned} \mu_k(z; \nu, m) &= \int_0^\infty x^{\nu+k} (x + z)^m e^{-x} dx \\ &= \Gamma(\nu + k + 1) z^{\nu+m+k+1} U(\nu + k + 1, \nu + m + k + 2, t) \\ &= \Gamma(\nu + k + 1) (-1)^m m! L_m^{(-\nu-m-k-1)}(z) \end{aligned}$$

with $L_n^{(\alpha)}(z)$ the **associated Laguerre polynomial**, since

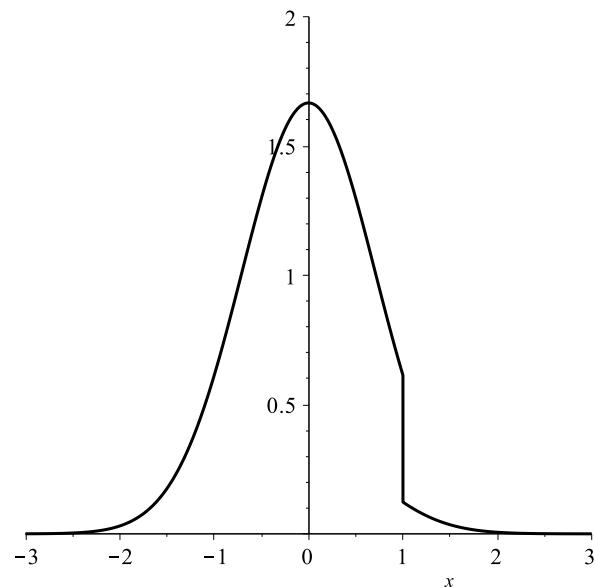
$$z^{\alpha+m} U(\alpha, \alpha + m + 1, z) = (-1)^m m! L_m^{(-\alpha-m)}(z), \quad m \in \mathbb{Z}^+$$

Discontinuous Weights

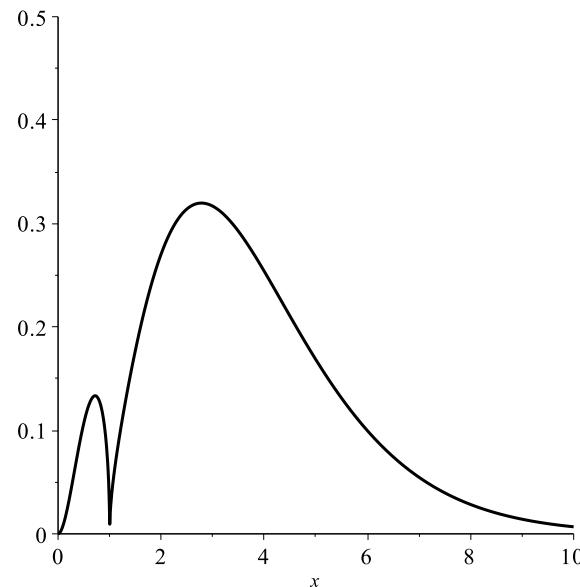
$$\omega_1(x; z) = \{1 + \xi - 2\xi\mathcal{H}(x - z)\} \exp(-x^2), \quad 0 < \xi < 1, \quad x, z \in \mathbb{R}$$

$$\omega_2(x; z) = \{1 - \xi\mathcal{H}(x - z)\} |x - z|^\lambda x^\alpha \exp(-x), \quad \nu, \lambda > 0, \quad x, z \in \mathbb{R}^+$$

where $\mathcal{H}(x)$ is the Heaviside step function.



$\omega_1(x; z)$



$\omega_2(x; z)$

Discontinuous Hermite weight

Consider the **discontinuous Hermite weight**

$$\omega_1(x; z) = \{1 + \xi - 2\xi\mathcal{H}(x - z)\} \exp(-x^2), \quad 0 < \xi < 1, \quad x, z \in \mathbb{R}$$

with $\mathcal{H}(x)$ the Heaviside step function. In this case

$$\mu_0(z) = \int_{-\infty}^{\infty} \omega(x; z) dx = \sqrt{\pi} [1 + \xi \operatorname{erf}(z)]$$

and define the Hankel determinant

$$\Delta_n(z) = \det \left[\mu_{j+k}(z) \right]_{j,k=0}^{n-1}$$

Then it can be shown that

$$S_n(z) = \frac{d}{dz} \ln \Delta_n(z)$$

satisfies

$$\left(\frac{d^2 S_n}{dz^2} \right)^2 - 4 \left(z \frac{d S_n}{dz} - S_n \right)^2 + 4 \left(\frac{d S_n}{dz} \right)^2 \left(\frac{d S_n}{dz} + 2n \right) = 0$$

which is S_{IV} , the P_{IV} σ -equation, with $(\vartheta_0, \vartheta_\infty) = (n, 0)$, and

$$w_n(z) = \frac{d}{dz} \ln \frac{\Delta_{n+1}(z)}{\Delta_n(z)} = S_{n+1}(z) - S_n(z)$$

satisfies P_{IV} with $(A, B) = (2n + 1, 0)$.

Discontinuous Laguerre weight

Consider the **discontinuous Laguerre weight**

$$\omega_2(x; z) = \{1 - \xi \mathcal{H}(x - z)\} |x - z|^\lambda x^\nu \exp(-x), \quad \nu, \lambda > 0, \quad x, z \in \mathbb{R}^+$$

with $\mathcal{H}(x)$ the Heaviside step function.

Since

$$\begin{aligned} \int_0^z x^\nu (z - x)^\lambda e^{-x} dx &= B(\lambda + 1, \nu + 1) z^{\nu + \lambda + 1} e^{-z} M(\lambda + 1, \nu + \lambda + 2, z) \\ \int_z^\infty x^\nu (x - z)^\lambda e^{-x} dx &= \Gamma(\lambda + 1) z^{\nu + \lambda + 1} e^{-z} U(\lambda + 1, \nu + \lambda + 2, z) \end{aligned}$$

with $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ the **Beta function**, and $M(a, b, z)$ and $U(a, b, z)$ the **Kummer functions**, then

$$\begin{aligned} \mu_0(z; \nu, \lambda) &= \int_0^\infty [1 - \xi \mathcal{H}(x - z)] x^\nu |x - z|^\lambda e^{-x} dx \\ &= \int_0^z x^\nu (z - x)^\lambda e^{-x} dx + (1 - \xi) \int_z^\infty x^\nu (x - z)^\lambda e^{-x} dx \\ &= z^{\nu + \lambda + 1} e^{-z} \{B(\lambda + 1, \nu + 1) M(\lambda + 1, \nu + \lambda + 2, z) \\ &\quad + (1 - \xi) \Gamma(\lambda + 1) U(\lambda + 1, \nu + \lambda + 2, z)\} \end{aligned}$$

Define the Hankel determinant

$$\Delta_n(z; \nu, \lambda) = \det \left[\mu_{j+k}(z; \nu, \lambda) \right]_{j,k=0}^{n-1}$$

then

$$H_n(z; \nu, \lambda) = z \frac{d}{dz} \ln \Delta_n(z; \nu, \lambda)$$

satisfies

$$\begin{aligned} z^2 \left(\frac{d^2 H_n}{dz^2} \right)^2 &= \left[(z + 2n + \nu + \lambda) \frac{dH_n}{dz} - H_n + (2n + 2\nu + \lambda)n \right]^2 \\ &\quad - 4 \left(\frac{dH_n}{dz} + n \right) \left(\frac{dH_n}{dz} + n + \nu \right) \left[z \frac{dH_n}{dz} - H_n + (n + \nu + \lambda)n \right] \end{aligned}$$

which is equivalent to a special case of S_V , the P_V σ -equation. Specifically, letting

$$H_n(z; \nu, \lambda) = \sigma - \frac{1}{4}(2n + \nu - \lambda)z + \frac{1}{2}n^2 + \frac{1}{2}n(\nu + \lambda) + \frac{1}{8}(\nu - \lambda)^2$$

yields

$$\left(z \frac{d^2 \sigma}{dz^2} \right)^2 - \left\{ 2 \left(\frac{d\sigma}{dz} \right)^2 - z \frac{d\sigma}{dz} + \sigma \right\}^2 + 4 \prod_{j=1}^4 \left(\frac{d\sigma}{dz} + \kappa_j \right) = 0 \quad S_V$$

with

$$\begin{aligned} \kappa_1 &= \frac{1}{2}n + \frac{3}{4}\nu + \frac{1}{4}\lambda, & \kappa_3 &= -\frac{1}{2}n - \frac{1}{4}\nu - \frac{3}{4}\lambda, \\ \kappa_2 &= \frac{1}{2}n - \frac{1}{4}\nu + \frac{1}{4}\lambda & \kappa_4 &= -\frac{1}{2}n - \frac{1}{4}\nu + \frac{1}{4}\lambda \end{aligned}$$

Discrete Orthogonal Polynomials

- P A Clarkson, “Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations”, *J. Phys. A* **46** (2013) 185205

Discrete Orthonormal Polynomials

Discrete orthonormal polynomials $\{p_n(x)\}$, $n = 0, 1, 2, \dots$, with respect to a discrete weight $\omega(k)$,

$$\sum_{k=0}^{\infty} p_m(k) p_n(k) \omega(k) = \delta_{m,n} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$$

satisfy the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$

The moments are

$$\mu_n = \sum_{k=0}^{\infty} k^n \omega(k), \quad n = 0, 1, 2, \dots$$

and the recurrence coefficients

$$a_n^2 = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad b_n = \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\tilde{\Delta}_n}{\Delta_n}$$

where

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix}, \quad \tilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}$$

In the case when the discrete weight has the form

$$\omega(k) = c(k)t^k, \quad t > 0$$

which is the case for the **Charlier polynomials** and **Meixner polynomials**, then

$$\mu_0 = \sum_{k=0}^{\infty} c(k)t^k \quad \Rightarrow \quad \mu_n = \sum_{k=0}^{\infty} k^n c(k)t^k = \delta^n(\mu_0), \quad \delta(\phi) = t \frac{d\phi}{dt}$$

Hence the determinants Δ_n and $\tilde{\Delta}_n$ are given by

$$\Delta_n(t) = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix} = \begin{vmatrix} \mu_0 & \delta(\mu_0) & \dots & \delta^{n-1}(\mu_0) \\ \delta(\mu_0) & \delta^2(\mu_0) & \dots & \delta^n(\mu_0) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{n-1}(\mu_0) & \delta^n(\mu_0) & \dots & \delta^{2n-2}(\mu_0) \end{vmatrix} =: \tilde{\mathcal{W}}_n(\mu_0)$$

$$\tilde{\Delta}_n(t) = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix} = \delta(\Delta_n) = \delta(\tilde{\mathcal{W}}_n(\mu_0))$$

and the recurrence coefficients are given by

$$a_n^2(t) = \delta^2(\ln \Delta_n) = \delta^2(\ln \tilde{\mathcal{W}}_n(\mu_0)), \quad b_n(t) = \delta\left(\ln \frac{\Delta_{n+1}}{\Delta_n}\right) = \delta\left(\ln \frac{\tilde{\mathcal{W}}_{n+1}(\mu_0)}{\tilde{\mathcal{W}}_n(\mu_0)}\right)$$

Discrete Pearson Equation

The **discrete Pearson equation** has the form

$$\Delta[\sigma(k)\omega(k)] = \tau(k)\omega(k)$$

where Δ is the forward difference operator

$$\Delta f(k) = f(k+1) - f(k)$$

- **Classical discrete orthogonal polynomials:** $\sigma(k)$ and $\tau(k)$ are polynomials with $\deg(\sigma) \leq 2$ and $\deg(\tau) = 1$

	$\omega(k)$	$\sigma(k)$	$\tau(k)$
Charlier	$\frac{t^k}{k!}$	k	$t - k$
Meixner	$\frac{(\alpha)_k t^k}{k!}$	k	$(t-1)k + t\alpha$

- **Semi-classical discrete orthogonal polynomials:** $\sigma(k)$ and $\tau(k)$ are polynomials either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$

	$\omega(k)$	$\sigma(k)$	$\tau(k)$
Generalized Charlier	$\frac{t^k}{(\beta)_k k!}$	$k(k+\beta-1)$	$-k^2 + (1-\beta)k + t$
Generalized Meixner	$\frac{(\alpha)_k t^k}{(\beta)_k k!}$	$k(k+\beta-1)$	$-k^2 + (1+t-\beta)k + t\alpha$

Generalized Charlier Polynomials

The **generalized Charlier polynomials** are orthogonal on \mathbb{N} with respect to the discrete weight

$$\omega(k) = \frac{t^k}{(\beta)_k k!}, \quad \beta > 0$$

with $(\beta)_k = \Gamma(\beta + k)/\Gamma(\beta)$ the Pochhammer symbol.

Lemma

For the generalized Charlier polynomials

$$\mu_0(t) = \sum_{k=0}^{\infty} \frac{t^k}{(\beta)_k k!} = \Gamma(\beta) t^{(1-\beta)/2} I_{\beta-1}(2\sqrt{t})$$

with $I_\nu(z)$ the modified Bessel function.

Corollary

The Hankel determinant $\Delta_n(t)$ is given by

$$\Delta_n(t) = \widetilde{\mathcal{W}}_n(\mu_0) = [\Gamma(\beta)]^n \widetilde{\mathcal{W}}_n\left(t^{(1-\beta)/2} I_{\beta-1}(2\sqrt{t})\right)$$

and the recurrence coefficients $a_n(t)$ and $b_n(t)$ have the form

$$a_n^2(t) = \left(t \frac{d}{dt}\right)^2 (\ln \Delta_n(t)), \quad b_n(t) = t \frac{d}{dt} \left(\ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}\right)$$

Theorem

(PAC [2013])

The function $S_n(t) = t \frac{d}{dt} \ln \Delta_n(t)$ where

$$\Delta_n(t) = [\Gamma(\beta)]^n \widetilde{\mathcal{W}}_n \left(t^{(1-\beta)/2} I_{\beta-1}(2\sqrt{t}) \right)$$

satisfies

$$\left[t \frac{d^2 S_n}{dt^2} \right]^2 = \left[n - (n + \beta - 1) \frac{dS_n}{dt} \right]^2 - 4 \frac{dS_n}{dt} \left[\frac{dS_n}{dt} - 1 \right] \left[t \frac{dS_n}{dt} - S_n + \frac{1}{2}n(n-1) \right]$$

which is equivalent to $S_{III'}$, the $P_{III'}$ σ -equation.

Proof. Making the transformation

$$S_n(t; \beta) = \sigma(t) + \frac{1}{2}t + \frac{1}{4}n^2 - \frac{1}{2}n(\beta + 1) - \frac{1}{4}\beta^2$$

yields

$$\left(t \frac{d^2 \sigma}{dt^2} \right)^2 + \left\{ 4 \left(\frac{d\sigma}{dt} \right)^2 - 1 \right\} \left(t \frac{d\sigma}{dt} - \sigma \right) + (n^2 - \beta^2) \frac{d\sigma}{dt} = \frac{1}{2}(n^2 + \beta^2)$$

which is the $P_{III'}$ σ -equation with parameters $(\theta_0, \theta_\infty) = (n + \beta, n - \beta)$.

Generalized Meixner polynomials

The **generalized Meixner polynomials** are orthogonal on \mathbb{N} with respect to the discrete weight

$$\omega(k) = \frac{(\alpha)_k t^k}{(\beta)_k k!}, \quad \alpha > 0, \quad \beta > 0$$

with $(\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$ the Pochhammer symbol.

Lemma

For the generalized Meixner polynomials

$$\mu_0(t) = \sum_{k=0}^{\infty} \frac{(\alpha)_k t^k}{(\beta)_k k!} = M(\alpha, \beta, t)$$

with $M(\alpha, \beta, t)$ the Kummer function.

Corollary

The Hankel determinant $\Delta_n(t)$ is given by

$$\Delta_n(t) = \widetilde{\mathcal{W}}_n(\mu_0) = \widetilde{\mathcal{W}}_n(M(\alpha, \beta, t))$$

and the recurrence coefficients $a_n(t)$ and $b_n(t)$ have the form

$$a_n^2(t) = \left(t \frac{d}{dt} \right)^2 (\ln \Delta_n(t)), \quad b_n(t) = t \frac{d}{dt} \left(\ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)} \right)$$

Theorem

(PAC [2013])

The function

$$S_n(t) = t \frac{d}{dt} \ln \widetilde{\mathcal{W}}_n(M(\alpha, \beta, t))$$

satisfies

$$\begin{aligned} \left(t \frac{d^2 S_n}{dt^2} \right)^2 &= \left[(t + n + \beta - 1) \frac{d S_n}{dt} - \frac{1}{2} n(n - 1 + 2\alpha) \right]^2 \\ &\quad - 4 \frac{d S_n}{dt} \left(\frac{d S_n}{dt} - n - \alpha + \beta \right) \left[t \frac{d S_n}{dt} - S_n + \frac{1}{2} n(n - 1) \right] \end{aligned}$$

which is equivalent to S_V , the P_V σ -equation.

Proof. Making the transformation

$$S_n(t) = \sigma(z) + \frac{1}{4}(2\alpha - \beta + 3n - 1)z + \frac{5}{8}n^2 + \frac{1}{4}(2\alpha - 3\beta - 1)n + \frac{1}{8}(2\alpha - \beta - 1)^2$$

with $z = t$, yields the P_V σ -equation

$$\left(z \frac{d^2 \sigma}{dz^2} \right)^2 - \left\{ 2 \left(\frac{d \sigma}{dz} \right)^2 - z \frac{d \sigma}{dz} + \sigma \right\}^2 + 4 \prod_{j=1}^4 \left(\frac{d \sigma}{dz} + \kappa_j \right) = 0 \quad S_V$$

with parameters

$$\kappa_1 = \frac{1}{4}(2\alpha - \beta - n - 1)$$

$$\kappa_3 = \frac{1}{4}(2\alpha - \beta + 3n - 1)$$

$$\kappa_2 = -\frac{1}{4}(2\alpha + \beta + n - 3)$$

$$\kappa_4 = -\frac{1}{4}(2\alpha - 3\beta + n + 1)$$

Conclusions

- The coefficients in the three-term recurrence relations associated with semi-classical generalizations of orthogonal polynomials and discrete orthogonal polynomials can often be expressed in terms of solutions of the Painlevé equations.
- These coefficients can be expressed as Hankel determinants which arise in the solution of the Painlevé equations and particularly the Painlevé σ -equations, the second-order, second-degree equations associated with the Hamiltonian representation of the Painlevé equations.
- These Hankel determinants arise in the special cases of the Painlevé equations when they have solutions in terms of the classical special functions, the “classical solutions” of the Painlevé equations.
- The moments of the semi-classical weights and the generalised discrete weights provide the link between the orthogonal polynomials and the associated Painlevé equation.
- These ideas can be extended to orthogonal polynomials in other contexts:
 - * on the unit circle (**PAC & Smith**); and
 - * on curves in the complex plane (**PAC, Loureiro & van Assche**).
- These results illustrate the increasing significance of the Painlevé equations in the field of orthogonal polynomials and special functions.