Extending Askey tableau by the inclusion of Krall and exceptional polynomials

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Universidad de Sevilla

18 de diciembre de 2014

(Work in progress)

Introduction: Classical and classical discrete orthogonal polynomials

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The physical meaning of the equations associated to these second order operators (specially to the differential ones) makes the corresponding families of orthogonal polynomials very useful in physics (specially in quantum physics).

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$$q_n(x) = L_n^\alpha(x) - \frac{1 + M(n+1)_\alpha}{1 + M(n)_\alpha} L_{n-1}^\alpha(x),$$

orthogonal with respect to

$$\mu_{M,\alpha} = \alpha\Gamma^2(\alpha) M\delta_0 + x^{\alpha-1}e^{-x}, \quad x > 0.$$

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This weight measure is a Geronimous transform of the Laguerre weight:

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$$-\left(\frac{d}{dx}\right)^2 + 2\left(x + \frac{\Omega'_F(x)}{\Omega_F(x)}\right)\frac{d}{dx} + 2\left(k + u_F - x\frac{\Omega'_F(x)}{\Omega_F(x)}\right) - \frac{\Omega''_F(x)}{\Omega_F(x)},$$

where

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$\Omega_F(x) \neq 0$, $x \in \mathbb{R}$ if and only if $\prod_{f\in F}(n-f) \geq 0$, $n \in \mathbb{N}$.

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Exceptional
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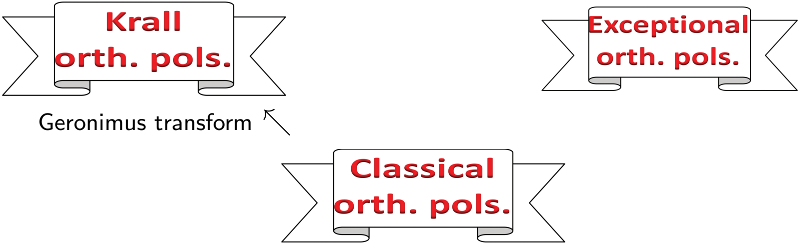
Krall orth. pols.
Exceptional orth. pols.
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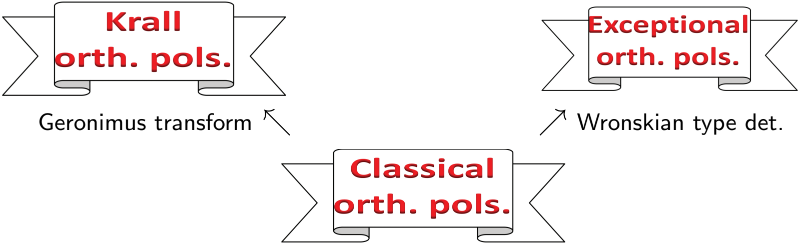
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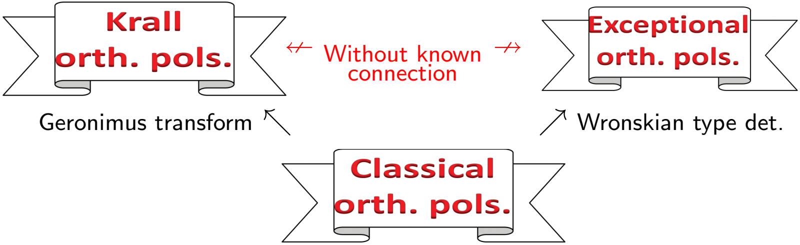
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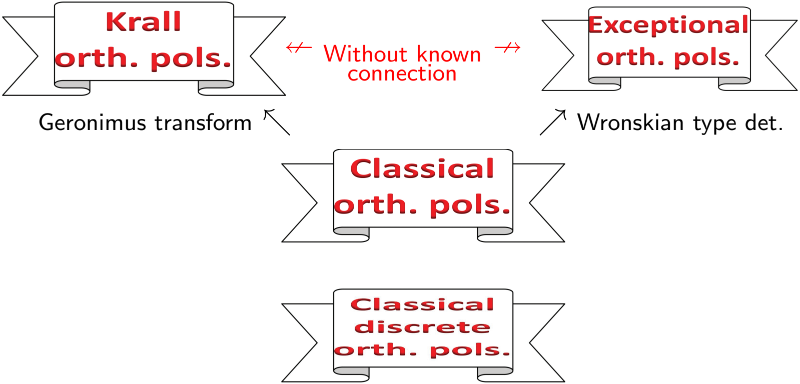
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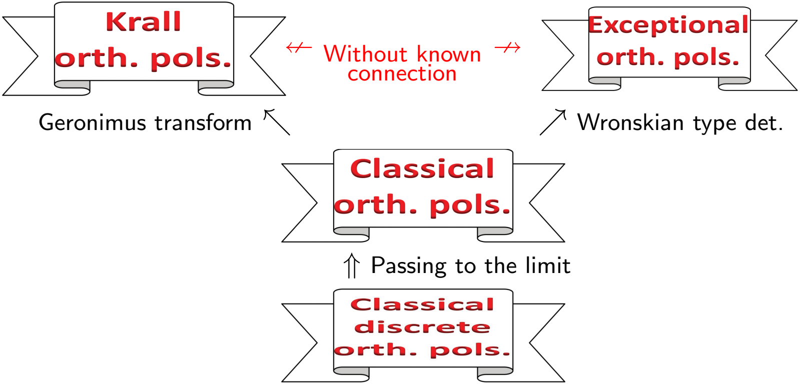
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In 2011, this speaker found the first examples of measures whose orthogonal polynomials are also eigenfunctions of higher order difference operator.

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Depending on the classical discrete measure, we have found several classes of suitable polynomials for which this procedure works (1 class for Charlier, 2 classes for Meixner and Krawtchouk and 4 classes for Hahn; the cross product of these classes also works).

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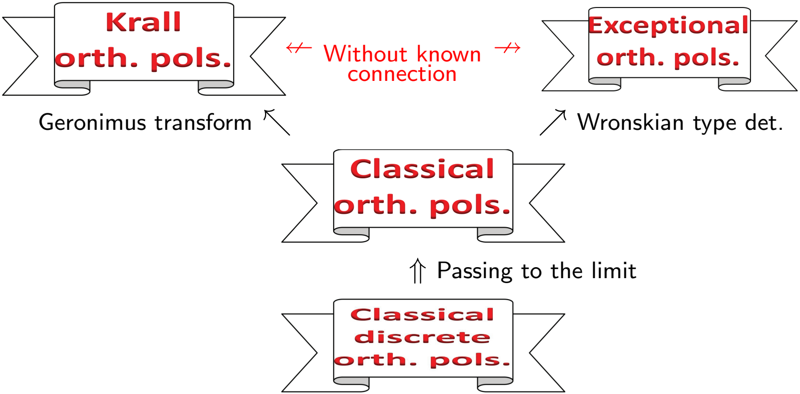
This allows a nice and important extension of Askey tableau.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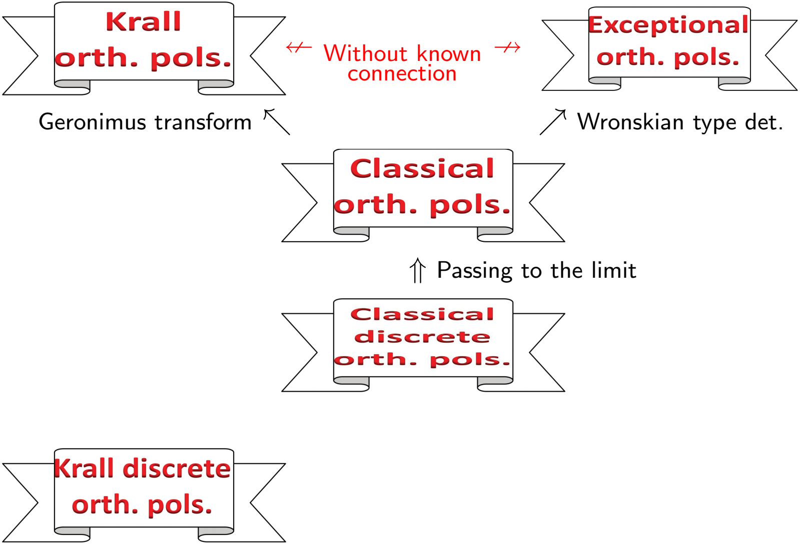

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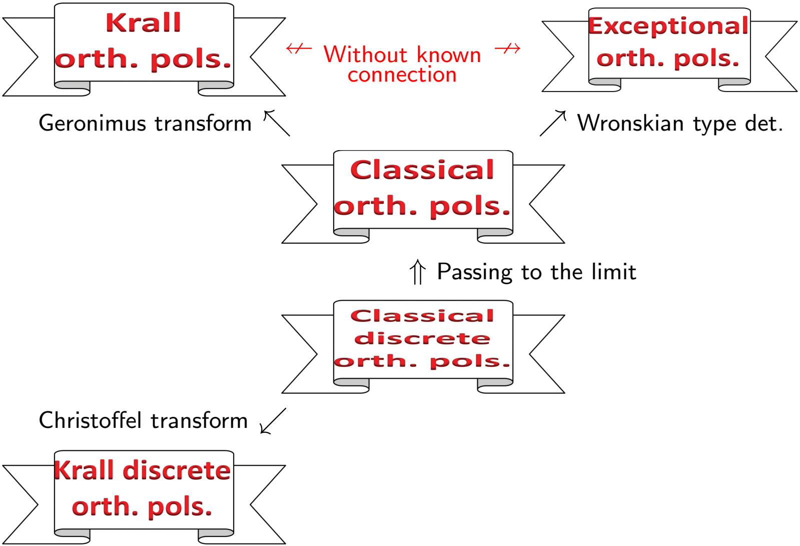

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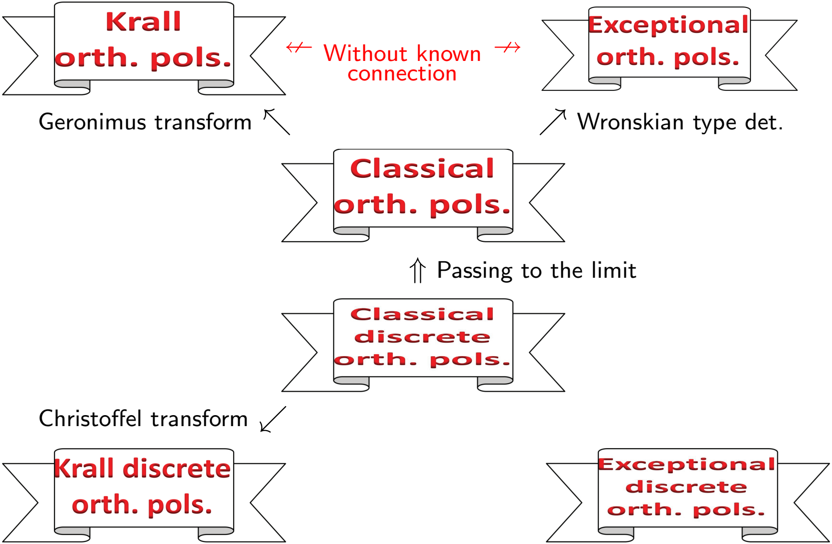

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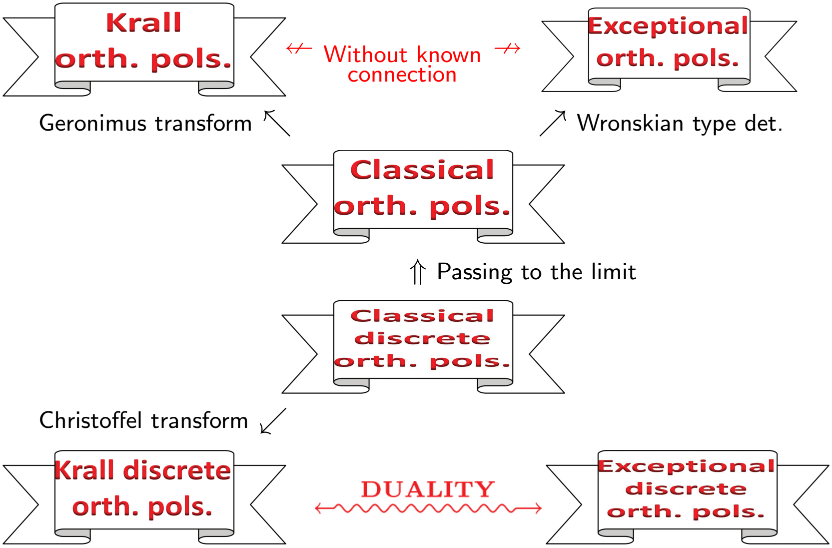

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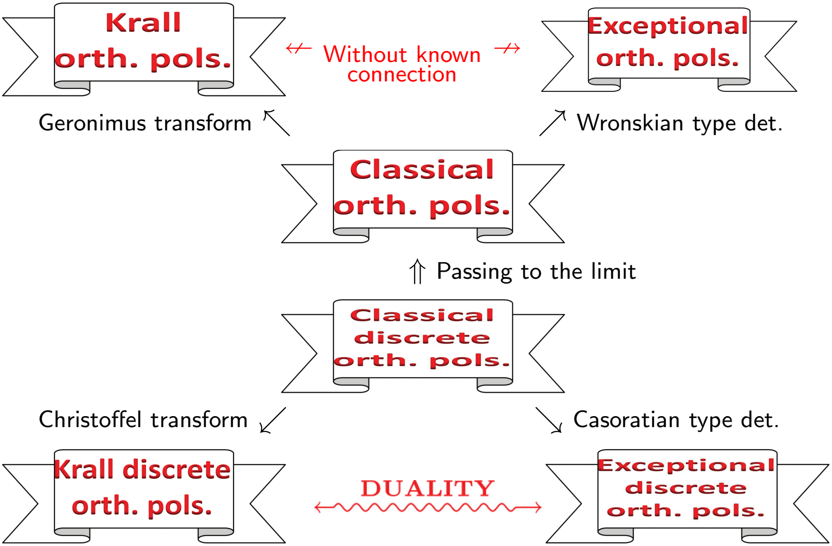
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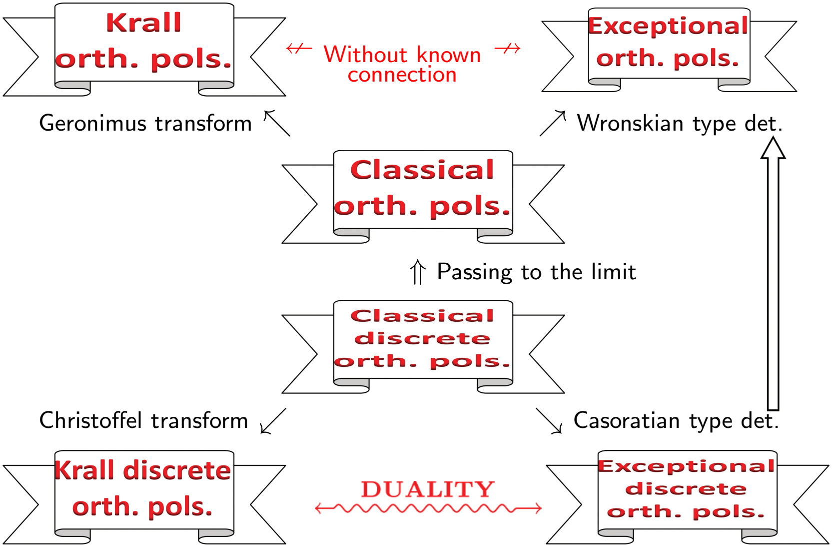
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Exceptional polynomials: Differential operators

The duality connection can be used to solve some of the most interesting questions concerning exceptional polynomials; for instance, to find necessary and sufficient conditions for the existence of orthogonality measures for exceptional polynomials.

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$\frac{e^{-x^2}}{\Omega_F^2(x)}$ if and only if $\prod_{f\in F}(n-f)\geq 0$, $n\in\mathbb{N}$ (Krein-Adler admissibility condition).

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Exceptional Hermite polynomials can be constructed starting from the Charlier-Krall polynomials, using duality and passing to the limit (as when one goes from Charlier to Hermite polynomials).

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Exceptional Hermite polynomials are orthogonal with respect to the weight $\frac{e^{-x^2}}{\Omega_F^2(x)}$ if and only if $\prod_{f\in F}(n-f)\geq 0$, $n\in\mathbb{N}$ (Krein-Adler admissibility condition).

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Using this approach we have solved the problem of determining admissibility conditions for Laguerre and Jacobi exceptional polynomials.

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where $V_F = \prod_{1\le i<v\le k}(f_j - f_i)$.