A q-generalization of the Bannai-Ito polynomials and the quantum superalgebra $\mathfrak{osp}_q(1|2)$

Luc Vinet

Centre de recherches mathématiques (CRM)
Université de Montréal

Work in collaboration with:
Vincent X. Genest (Montreal) and A. Zhedanov (Donetsk)

- Polynomials $B_n(x)$ introduced by Bannai and Ito in their 1984 book "*Algebraic Combinatorics I: Association Schemes*"
- Depend on four real parameters ρ_1, ρ_2, r_1, r_2
- For n even, $B_n(x)$ is of the form

$$B_n(x) \sim {}_4F_3\left(\begin{matrix} -\frac{n}{2}, \frac{n+1}{2}+h, x-r_1+1/2, -x-r_1+1/2 \\ 1-r_1-r_2, \rho_1-r_1+\frac{1}{2}, \rho_2-r_1+\frac{1}{2} \end{matrix} \middle| 1\right)$$
$$+ \frac{(\frac{n}{2})(x-r_1+\frac{1}{2})}{(\rho_1-r_1+\frac{1}{2})(\rho_2-r_1+\frac{1}{2})} \times {}_4F_3\left(\begin{matrix} 1-\frac{n}{2}, \frac{n+1}{2}+h, x-r_1+3/2, -x-r_1+1/2 \\ 1-r_1-r_2, \rho_1-r_1+\frac{3}{2}, \rho_2-r_1+\frac{3}{2} \end{matrix} \middle| 1\right)$$

where $h = \rho_1+\rho_2-r_1-r_2+1/2$

- For n odd, a similar expression

- $B_n(x)$ are invariant under $\rho_1 \leftrightarrow \rho_2$ and $r_1 \leftrightarrow r_2$
- Identified by Bannai and Ito as $q \to -1$ limits of q-Racah OPs
- $B_n(x)$ satisfy the 3-term recurrence relation

$$xB_n(x) = B_{n+1}(x) + (\rho_1 - A_n - C_n)B_n(x) + A_{n-1}C_n B_{n-1}(x)$$

with recurrence coefficients

$$A_n = \begin{cases} \frac{(n+1+2\rho_1-2r_1)(n+1+2\rho_1-2r_2)}{4(n+\rho_1+\rho_2-r_1-r_2+1)} & n \text{ even} \\ \frac{(n+1+2\rho_1+2\rho_2-2r_1-2r_2)(n+1+2\rho_1+2\rho_2)}{4(n+\rho_1+\rho_2-r_1-r_2+1)} & n \text{ odd} \end{cases}$$

$$C_n = \begin{cases} -\frac{n(n-2r_1-2r_2)}{4(n+\rho_1+\rho_2-r_1-r_2)} & n \text{ even} \\ -\frac{(n+2\rho_2-2r_2)(n+2\rho_2-2r_1)}{4(n+\rho_1+\rho_2-r_1-r_2)} & n \text{ odd} \end{cases}$$

• Positivity condition $A_{n-1}C_n > 0$ satisfied only for $n = 1, \dots, N$

• Hence $B_n(x)$ orthogonal w.r.t. discrete positive measure ω_s

$$\sum_{s=0}^{N} \omega_s B_n(x_s) B_m(x_s) = h_n \delta_{mn}$$

where grid points x_s have the expression

$$x_s = (-1)^s(s/2 + a + 1/4) - 1/4 \qquad s = 0, 1, \dots, N$$

• The Bannai–Ito polynomials are *bispectral*, i.e. in addition to 3-term recurrence relation, they satisfy an eigenvalue equation of the form

$$\mathscr{L} B_n(x) = \lambda_n B_n(x)$$

where $B_n(x)$ is a first-order operator in shifts and reflections

- The operator $\mathscr{L}$ reads

$$\mathscr{L} = F(x)\,[R-1] + G(x)\,[T^{+}R-1] + h$$

where $h = \rho_1 + \rho_2 - r_1 - r_2 + 1/2$ and

$$F(x) = -\frac{(x-\rho_1)(x-\rho_2)}{x} \qquad G(x) = \frac{(x-r_1+1/2)(x-r_2+1/2)}{x+1/2}$$

and $Rf(x) = f(-x)$, $T^{+}f(x) = f(x+1)$

- The eigenvalues are

$$\lambda_n = (-1)^n(n+h) \qquad n = 0,1,2,\ldots$$

- $B_n(x)$ satisfy the Leonard duality property since $\mathscr{L}$ is tridiagonal

$$\mathscr{L}\, f(x_s) = \begin{cases} G(x_s)f(x_{s+1}) + F(x_s)f(x_{s-1}) - [G(x_s)+F(x_s)-h]\,f(x_s) & s \text{ even} \\ F(x_s)f(x_{s+1}) + G(x_s)f(x_{s-1}) - [G(x_s)+F(x_s)-h]\,f(x_s) & s \text{ odd} \end{cases}$$

on the grid x_s

2. Algebraic interpretation of the BI polynomials

2.1 Bannai–Ito algebra

• The bispectral properties of the Bannai–Ito polynomials are encoded in an algebraic structure called the Bannai–Ito algebra

• Define

$$K_1 = \mathscr{L} = F(x)\,[R-1] + G(x)\,[T^{+}R-1] + h$$

$$K_2 = 2x + 1/2$$

• Introduce an operator K_3 as follows

$$K_3 = \{K_1, K_2\} - 4(\rho_1\rho_2 - r_1 r_2)$$

where $\{A,B\} = AB + BA$

• Operators K_1, K_2, K_3 generate the Bannai-Ito algebra

$$\{K_1, K_2\} = K_3 + \omega_3 \qquad \{K_2, K_3\} = K_1 + \omega_1 \qquad \{K_3, K_1\} = K_2 + \omega_2$$

- The structure constants ω_1, ω_2, ω_3 read

$$\omega_1 = 4(\rho_1\rho_2 + r_1 r_2) \quad \omega_2 = 2(\rho_1^2 + \rho_2^2 - r_1^2 - r_2^2) \quad \omega_3 = 4(\rho_1\rho_2 - r_1 r_2)$$

- The BI algebra has a Casimir operator

$$Q_{BI} = K_1^2 + K_2^2 + K_3^2$$

which commutes with all generators

- In this realization it takes the value

$$Q_{BI} = 2(\rho_1^2 + \rho_2^2 + r_1^2 + r_2^2) - 1/4$$

- BI polynomials play a double role in this context
  - $B_n(x)$ are basis vectors for irreps. of the BI algebra
  - $B_n(x)$ are transition coefficients between K_1- and K_2- eigenbases

2.2 Racah coefficients for Lie superalgebra $\mathfrak{osp}(1|2)$

- $\mathfrak{osp}(1|2)$ superalgebra is a $\mathbb{Z}$-graded algebra generated by A_0 and $A_\pm$
- A_0 is an even generator and $A_\pm$ are odd generators
- The product is graded

$$\{A,B\} = AB + BA \qquad \text{if both } A \text{ and } B \text{ are odd}$$
$$[A,B] = AB - BA \qquad \text{otherwise}$$

- Defining relations

$$[A_0, A_\pm] = \pm A_\pm \qquad \{A_+, A_-\} = 2A_0$$

- The Lie subalgebra $\mathfrak{sp}(2) \simeq \mathfrak{su}(1,1)$ is generated by the even elements

$$F_\pm = A_\pm^2/2 \qquad \text{and} \qquad A_0$$

- The grade involution P (P^2=1) can be used to distinguish even and odd elements

$$\{A_\pm, P\} = 0 \qquad [A_0, P] = 0 \qquad [F_\pm, P] = 0$$

- $\mathfrak{osp}(1|2)$ can thus be presented in terms of the relations

$$[A_0, A_\pm] = \pm A_\pm \quad \{A_+, A_-\} = 2A_0 \quad [A_0, P] = \{A_\pm, P\} = 0 \quad P^2 = 1$$

- This presentation has been referred to as $sl_{-1}(2)$ as it corresponds to $q \to -1$ limit of $sl_q(2)$
- The Casimir reads

$$\mathcal{Q} = A_+A_-P - A_0P + P/2$$

- Corresponds to sCasimir multiplied by the involution
- Coalgebra structure $\Delta : \mathfrak{osp}(1|2) \to \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2)$

$$\Delta(A_0) = A_0 \otimes 1 + 1 \otimes A_0 \quad \Delta(A_\pm) = A_\pm \otimes P + 1 \otimes A_\pm \quad \Delta(P) = P \otimes P$$

Let $V^{(\lambda_i)}$ be an irrep of $\mathfrak{osp}(1|2)$ specified by value λ_i of $\mathcal{Q}$

- Δ allows to construct direct product of representations

The Racah problem arises in the coupling of three irreps. and in the construction of bases for $V = V^{(\lambda_1)} \otimes V^{(\lambda_2)} \otimes V^{(\lambda_3)}$

- First basis corresponds to coupling $V^{(\lambda_1)} \otimes V^{(\lambda_2)}$, then $V^{(\lambda_3)}$

  Basis vectors $|q_{12};\Lambda\rangle$ are joint eigenvectors of

  Intermediate Casimir $\mathscr{Q}_{12} = \Delta(\mathscr{Q}) \otimes 1$ with eigenvalue q_{12}

  Total Casimir $\mathscr{Q} = (1 \otimes \Delta)\Delta(\mathscr{Q})$ with eigenvalue Λ

- Second basis corresponds to coupling $V^{(\lambda_2)} \otimes V^{(\lambda_3)}$, then $V^{(\lambda_1)}$

  Basis vectors $|q_{23};\Lambda\rangle$ are joint eigenvectors of

  Intermediate Casimir $\mathscr{Q}_{23} = 1 \otimes \Delta(\mathscr{Q})$ with eigenvalue q_{23}

  Total Casimir $\mathscr{Q} = (1 \otimes \Delta)\Delta(\mathscr{Q})$ with eigenvalue Λ

Racah coefficients are the transition coefficients between these bases

$$R^{\lambda_1,\lambda_2,\lambda_3}_{q_{12},q_{23},\Lambda} = \langle q_{12};\Lambda|q_{23};\Lambda\rangle$$

- The key observation is that upon taking

$$K_3 = -\mathcal{Q}_{12} \qquad K_1 = -\mathcal{Q}_{23}$$

one finds the Bannai–Ito algebra

$$\{K_1, K_2\} = K_3 + 2(\lambda_1\lambda_2 + \lambda_3\Lambda)$$
$$\{K_2, K_3\} = K_1 + 2(\lambda_2\lambda_3 + \lambda_1\Lambda)$$
$$\{K_3, K_1\} = K_2 + 2(\lambda_1\lambda_3 + \lambda_2\Lambda)$$

where K_2 is defined by the third relation

- The BI Casimir operator takes the value

$$Q_{BI} = K_1^2 + K_2^2 + K_3^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \Lambda^2 - 1/4$$

- Construction of corresponding irreps. of the BI algebra yields

$$R^{\lambda_1,\lambda_2,\lambda_3}_{q_{12},q_{23},\Lambda} \propto \text{Bannai–Ito OPs}$$

3. Outline of the remainder of the talk

- Is it possible to q-generalize the Bannai–Ito polynomials ?
- One (fruitful) idea is exploit the passage

$$\mathfrak{osp}(1|2) \longrightarrow \mathfrak{osp}_q(1|2)$$

$$\text{superalgebra} \longrightarrow \text{quantum superalgebra}$$

- We shall hence consider the Racah problem for $\mathfrak{osp}_q(1|2)$
- This will lead to the q-generalization of the BI OPs
- Algebraic picture will give numerous properties
- ...and a somewhat surprising result !

4. The quantum superalgebra $\mathfrak{osp}_q(1|2)$

- $\mathfrak{osp}_q(1|2)$ is generated by $A_0, A_\pm, P$ with

$$[A_0, A_\pm] = \pm A_\pm \quad \{A_+, A_-\} = \frac{q^{A_0} - q^{-A_0}}{q^{1/2} - q^{-1/2}} \quad [A_0, P] = \{A_\pm, P\} = 0$$

- Again P with $P^2 = 1$ is the grade involution
- With $K = q^{A_0}$, $K^{-1} = q^{-A_0}$ relations are

$$KA_+ = qA_+K \qquad KA_- = q^{-1}A_-K \qquad KK^{-1} = 1 \qquad P^2 = 1$$

$$\{A_+, A_-\} = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}} \qquad \{A_\pm, K\} = [A_0, K] = 0$$

- This algebra admits the Casimir operator

$$C = \left[A_+A_- - \left(\frac{q^{-1/2}}{q - q^{-1}}\right)K + \left(\frac{q^{1/2}}{q - q^{-1}}\right)K^{-1}\right]P$$

which commutes with all generators

- The algebra $\mathfrak{osp}_q(1|2)$ has a coalgebra structure under the coproduct Δ

$$\Delta(A_\pm) = A_\pm \otimes K^{1/2} P + K^{-1/2} \otimes A_\pm \qquad \Delta(P) = P \otimes P \qquad \Delta(K) = K \otimes K$$

Representations

- Let $V^{(\epsilon,\mu)}$ be the vector space with basis $e_n^{(\epsilon,\mu)}$ for $n = 0, 1, 2, \ldots$
- Take $\epsilon = \pm 1$ and $\mu > -1/2$ and endow $V^{(\epsilon,\mu)}$ with the actions

$$K e_n^{(\epsilon,\mu)} = q^{n+\mu+1/2} e_n^{(\epsilon,\mu)} \qquad P e_n^{(\epsilon,\mu)} = \epsilon(-1)^n e_n^{(\epsilon,\mu)}$$
$$A_+ e_n^{(\epsilon,\mu)} = \sqrt{\sigma_{n+1}}\, e_{n+1}^{(\epsilon,\mu)} \qquad A_- e_n^{(\epsilon,\mu)} = \sqrt{\sigma_n}\, e_{n-1}^{(\epsilon,\mu)}$$

where

$$\sigma_n = \begin{cases} \left(\dfrac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}\right)\left(\dfrac{q^{(n+2\mu)/2} + q^{-(n+2\mu)/2}}{q^{1/2} + q^{-1/2}}\right) & n \text{ even} \\[2ex] \left(\dfrac{q^{n/2} + q^{-n/2}}{q^{1/2} + q^{-1/2}}\right)\left(\dfrac{q^{(n+2\mu)/2} - q^{-(n+2\mu)/2}}{q^{1/2} - q^{-1/2}}\right) & n \text{ odd} \end{cases}$$

On $V^{(\epsilon,\mu)}$ we have $A_\pm^\dagger = A_\mp$

- On $V^{(\epsilon,\mu)}$ the Casimir operator C has the action

$$C e_n^{(\epsilon,\mu)} = -\epsilon \left(\frac{q^{\mu} - q^{-\mu}}{q - q^{-1}} \right) e_n^{(\epsilon,\mu)}$$

- $V^{(\epsilon,\mu)}$ has a realization on monomials:

$$e_n^{(\epsilon,\mu)}(y) = \frac{y^n}{\sqrt{\sigma_1 \sigma_2 \cdots \sigma_n}}$$

$$A_+ = y \qquad A_- = \frac{q^{\mu}}{(q - q^{-1})y}(T_q^+ - R_y) - \frac{q^{-\mu}}{(q - q^{-1})y}(T_q^- - R_y)$$

$$K = q^{\mu+1/2} T_q \qquad P = \epsilon R_y,$$

where

$$T_q^{\pm} f(y) = f(q^{\pm 1} y) \qquad R_y f(y) = f(-y)$$

- In the $q \to 1$ limit, A_- tends to the Dunkl derivative

$$A_- \longrightarrow \frac{\partial}{\partial y} + \frac{\mu}{y}(1 - R_y)$$

5. The Racah problem for $\mathfrak{osp}_q(1|2)$ and a q-analog of the Bannai–Ito algebra

- Let $V = V^{(\epsilon_1,\mu_1)} \otimes V^{(\epsilon_2,\mu_2)} \otimes V^{(\epsilon_3,\mu_3)}$
- Two natural bases in decomposing V in irreducible components:
  - Basis $|c_{12};c\rangle$ corresponding to diagonalization of

$$C_{12} = \Delta(C) \otimes 1 \quad \text{and} \quad C = (1 \otimes \Delta)\Delta(C) = (\Delta \otimes 1)\Delta(C)$$

$$C_{12}|c_{12};c\rangle = c_{12}|c_{12};c\rangle \quad C|c_{12};c\rangle = c|c_{12};c\rangle$$

  - Basis $|c_{23};c\rangle$ corresponding to diagonalization of

$$C_{23} = 1 \otimes \Delta(C) \quad \text{and} \quad C = (1 \otimes \Delta)\Delta(C)$$

$$C_{23}|c_{23};c\rangle = c_{23}|c_{23};c\rangle \quad C|c_{23};c\rangle = c|c_{23};c\rangle$$

- The Racah coefficients are defined as

$$R^{\mu_1,\mu_2,\mu_3}_{c_{12},c_{23},c} = \langle c_{23};c|c_{12};c\rangle$$

- Introduce the operators

$$I_3 = -C_{12} \qquad I_1 = -C_{23}$$

- Define the q-anticommutator

$$\{A, B\}_q = q^{1/2} AB + q^{-1/2} BA$$

- One finds

$$\{I_1, I_2\}_q = I_3 + \delta_3 \quad \{I_2, I_3\}_q = I_1 + \delta_1 \quad \{I_3, I_1\}_q = I_2 + \delta_2 \qquad (\star)$$

where

$$\delta_1 = (q^{1/2} + q^{-1/2})(\tau_1 \tau + \tau_2 \tau_3)$$
$$\delta_2 = (q^{1/2} + q^{-1/2})(\tau_2 \tau + \tau_3 \tau_1)$$
$$\delta_3 = (q^{1/2} + q^{-1/2})(\tau_3 \tau + \tau_1 \tau_2)$$

- The parameters τ_1, τ_2, τ_3 and τ are determined by the representations

- One has

$$\tau_i = \epsilon_i \left(\frac{q^{\mu_i} - q^{-\mu_i}}{q - q^{-1}} \right) \quad i = 1,2,3 \quad \text{and} \quad \tau = -c = \epsilon \left(\frac{q^{\mu} - q^{-\mu}}{q - q^{-1}} \right)$$

where ϵ and μ need to be determined

- The q-Bannai–Ito algebra ($\star$) has for Casimir operator

$$\mathscr{C} = (q^{-1/2} - q^{3/2}) I_1 I_2 I_3 + q I_1^2 + q^{-1} I_2^2 + q I_3^2 \\ - (1-q)\delta_1 I_1 - (1-q^{-1})\delta_2 I_2 - (1-q)\delta_3 I_3$$

- In the present realization it takes the value

$$\mathscr{C} = \tau_1^2 + \tau_2^2 + \tau_3^2 + \tau^2 - \frac{q}{(1+q)^2} - (q-q^{-1})^2 \tau_1 \tau_2 \tau_3 \tau$$

- In the $q \to 1$ limit, ($\star$) goes to the Bannai–Ito algebra
- We shall take $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ for simplicity

- Structure of $V^{(\epsilon,\mu)}$ gives the spectra of I_1, I_3 and C
- Eigenvalues λ_k of I_3 are

$$\lambda_k = (-1)^k \left(\frac{q^{k+\mu_1+\mu_2+1/2} - q^{-(k+\mu_1+\mu_2+1/2)}}{q-q^{-1}} \right) \qquad k = 0,1,\ldots,N$$

- Eigenvalues ϑ_s of I_1 are

$$\vartheta_s = (-1)^s \left(\frac{q^{s+\mu_2+\mu_3+1/2} - q^{-(s+\mu_2+\mu_3+1/2)}}{q-q^{-1}} \right) \qquad s = 0,1,\ldots,N$$

- Eigenvalues of C (total Casimir) are

$$c_N = (-1)^{N+1} \left(\frac{q^{N+\mu_1+\mu_2+\mu_3+1} - q^{-(N+\mu_1+\mu_2+\mu_3+1)}}{q-q^{-1}} \right),$$

where the integer N is determined by

$$\mu = \mu_N = \mu_1+\mu_2+\mu_3+N+1$$

- Since $I_1 = -C_{23}$, $I_3 = -C_{12}$, rewrite

$$|c_{12};c\rangle \to |k;N\rangle \qquad |c_{23};c\rangle \to |s;N\rangle$$

- We have

$$I_3\,|k;N\rangle = \lambda_k\,|k;N\rangle \qquad C\,|k;N\rangle = c_N\,|k;N\rangle$$

and

$$I_1\,|s;N\rangle = \vartheta_s\,|s;N\rangle \qquad C\,|s;N\rangle = c_N\,|s;N\rangle$$

- The Racah coefficients are

$$R_{k;N}(s) = \langle s;N|k;N\rangle$$

- We can now determine the $(N+1)\times(N+1)$ matrices representing I_1, I_2 and I_3 in the $|k;N\rangle$-basis that verify the $(\star)$ relations

- One finds that

$$I_1|k;N\rangle = U_{k+1}\,|k+1;N\rangle + V_k\,|k;N\rangle + U_k\,|k-1;N\rangle$$

with

$$V_k = \left[\left(\frac{a-a^{-1}}{q-q^{-1}}\right) - A_k - C_k\right] \qquad U_k = \sqrt{A_{k-1}C_k}$$

- The coefficients read

$$A_k = -\frac{(1+abp^k)(1-acp^k)(1-adp^k)(1-abcdp^{k-1})}{a(q-q^{-1})(1-abcdp^{2k-1})(1-abcdp^{2k})}$$

$$C_k = \frac{a(1-p^k)(1-bcp^{k-1})(1-bdp^{k-1})(1+cdp^{k-1})}{(q-q^{-1})(1-abcdp^{2k-2})(1-abcdp^{2k-1})}$$

where $p = -q$ and

$$a = q^{\mu_2+\mu_3+1/2} \qquad b = (-1)^{N+1}q^{\mu_1-\mu_N+1/2}$$
$$c = (-1)^{N+1}q^{\mu_1+\mu_N+1/2} \qquad d = q^{\mu_2-\mu_3+1/2}$$

- Consider the overlap $\langle s;N|\, I_1 \,|k;N\rangle$
- Since I_1 is diagonal on $|s;N\rangle$

$$\langle s;N|\, I_1 \,|k;N\rangle = \vartheta_s \langle s;N|k;N\rangle = \vartheta_s R_{k;N}(\vartheta_s)$$

Since I_1 is tridiagonal on $|k;N\rangle$

$$\langle s;N|\, I_1 \,|k;N\rangle = U_{k+1} R_{k+1;N}(\vartheta_s) + V_k R_{k;N}(\vartheta_s) + U_k R_{k-1;N}(\vartheta_s)$$

- Write $R_{k;N}(\vartheta_s) = \langle s;N|0;N\rangle\, P_k(\vartheta_s)$ with $P_0(\vartheta_s) \equiv 1$
- $P_k(\vartheta_s)$ satisfy the three-term recurrence relation

$$\vartheta_s P_k(\vartheta_s) = U_{k+1} P_{k+1}(\vartheta_s) + V_k P_k(\vartheta_s) + U_k P_{k-1}(\vartheta_s),$$

in the variable $\vartheta_s = (-1)^s \left(\frac{q^{s+\mu_2+\mu_3+1/2} - q^{-(s+\mu_2+\mu_3+1/2)}}{q - q^{-1}} \right)$

- Truncation $U_{N+1} = 0$ and positivity $U_k > 0$ for $k = 1, \ldots, N$ are satisfied
- $P_k(\vartheta_s)$ are positive-definite OPs

- Upon taking

$$P_k(\vartheta_s) = \sqrt{\frac{A_0 A_1 \cdots A_{k-1}}{C_1 C_2 \cdots C_k}}\, \widehat{P}_k(\vartheta_s)$$

and setting $p = -q$, one finds that

$$\widehat{P}_k(\vartheta_s) = {}_4\varphi_3\left(\begin{matrix} p^{-k}, q^{2\mu_1+2\mu_2} p^{k+1}, q^{2\mu_2+2\mu_3} p^{s+1}, p^{-s} \\ p^{-N}, q^{2\mu_1+2\mu_2+2\mu_3+1} p^{N+1}, q^{2\mu_2+1} \end{matrix} \middle| p; p\right)$$

- These correspond to p-Racah OPs
- The p-Racah weight at $p = -q$ gives the orthogonality of the q-analogs of the BI polynomials
- From the explicit expression, Leonard duality property follows
- The $q \to 1$ limit leads to Bannai-Ito polynomials

- Not imposing the truncation condition, the q Bannai–Ito OPs can be presented in the following way

$$P_n(x) = {}_4\varphi_3\left(\begin{matrix} p^{-n},\, abcd\,p^{n-1},\, -az,\, az^{-1} \\ -ab,\, ac,\, ad \end{matrix}\,\middle|\, p; p\right) \qquad x = z - z^{-1}$$

with $p = -q$

- They satisfy the eigenvalue equation

$$\mathscr{L} P_n(x) = (1 - (-q)^{-n})(1 - abcd(-q)^{n-1})\, P_n(z)$$

where

$$\mathscr{L} = B(z)\,[T_q \mathscr{I} - 1] + B(-z^{-1})[T_q^{-1} \mathscr{I} - 1]$$

with $\mathscr{I} f(z) = f(z^{-1})$ and

$$B(z) = \frac{(1+az)(1+bz)(1-cz)(1-dz)}{(1+z^2)(1-qz^2)}$$

- Upon defining

$$I_1 = \alpha_1 \mathscr{L} + \beta_1$$

and

$$I_2 = \alpha_2(z - z^{-1}) + \beta_2$$

for some parameters α_i, β_i, one can realize the q-Bannai–Ito algebra

$$\{I_1, I_2\}_q = I_3 + \delta_3 \quad \{I_2, I_3\}_q = I_1 + \delta_1 \quad \{I_3, I_1\}_q = I_2 + \delta_2$$

- Again Casimir operator

$$\begin{aligned}\mathscr{C} = (q^{-1/2} - q^{3/2}) I_1 I_2 I_3 + q I_1^2 + q^{-1} I_2^2 + q I_3^2 \\ - (1-q)\delta_1 I_1 - (1-q^{-1})\delta_2 I_2 - (1-q)\delta_3 I_3\end{aligned}$$

takes a definite value (involving parameters a, b, c, d, q)

- Algebra relations can be used to derive the recurrence relation

- The recurrence relation is

$$x P_n(x) = B_n P_{n+1}(x) + [a - a^{-1} - B_n - D_n] P_n(x) + D_n P_{n-1}(x)$$

where

$$B_n = -\frac{(1+abp^k)(1-acp^k)(1-adp^k)(1-abcdp^{k-1})}{a(1-abcdp^{2k-1})(1-abcdp^{2k})}$$

$$D_n = \frac{a(1-p^k)(1-bcp^{k-1})(1-bdp^{k-1})(1+cdp^{k-1})}{(1-abcdp^{2k-2})(1-abcdp^{2k-1})}$$

with $p = -q$

- From these formulas, it is observed that the q-Bannai–Ito OPs can formally be obtained from the Askey-Wilson OPs through

$$z \to -iz \quad a \to -ia \quad b \to -ib \quad c \to ic \quad d \to id$$

and taking $q \to -q$

7. Conclusion

- Bannai–Ito polynomials have q-analogs
- Natural path: consider Racah problem for $\mathfrak{osp}_q(1|2)$
- Leads to q-deformation of the BI algebra
- Representations give the 3-term recurrence relation
- When $q \to 1$, BI OPs are recovered from their q-analogs
- The OPs can formally obtained from the AW with $q \to -q$ and complexification of parameters and variable

Outlook

- Continuous orthogonality relation for q-Bannai-Ito ?
- Limits and special cases
- q-deformed Dunkl oscillator model